WEYL-TITCHMARSH M-FUNCTION ASYMPTOTICS, LOCAL UNIQUENESS RESULTS, TRACE FORMULAS, AND BORG-TYPE THEOREMS FOR DIRAC OPERATORS

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Dedicated to F. V. Atkinson, one of the pioneers of this subject.

Abstract. We explicitly determine the high-energy asymptotics for Weyl-Titchmarsh matrices associated with general Dirac-type operators on half-lines and on $\mathbb{R}$. We also prove new local uniqueness results for Dirac-type operators in terms of exponentially small differences of Weyl-Titchmarsh matrices. As concrete applications of the asymptotic high-energy expansion we derive a trace formula for Dirac operators and use it to prove a Borg-type theorem.

1. Introduction

While the high-energy asymptotics, $|z| \to \infty$, of scalar-valued Weyl-Titchmarsh functions, $m_+(z, x_0)$, associated with general half-line Dirac-type differential expressions of the form

$$J\frac{d}{dx} - B(x), \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.1}$$

and B a self-adjoint 2×2 matrix with real-valued coefficients, $B^{(n)} \in L^1([x_0, c])^{2\times 2}$ for some $n \in \mathbb{N}_0 (= \mathbb{N} \cup \{0\})$ and all $c > x_0$, received some attention over the past two decades as can be inferred, for instance, from [32], [56], [60], [61], [97] (and the literature therein), it may perhaps come as a surprise that the corresponding matrix extension of this problem, considering general matrix-valued differential expressions of the type

$$J\frac{d}{dx} - B(x), \quad J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \tag{1.2}$$

with I_m the identity matrix in $\mathbb{C}^m$, $m \in \mathbb{N}$, and B a self-adjoint $2m \times 2m$ matrix satisfying $B^{(n)} \in L^1([x_0, c])^{2m\times 2m}$ for some $n \in \mathbb{N}_0$ and all $c > x_0$, apparently, received no attention at all. (It should be noted that this observation discounts papers in the special scattering theoretic case concerned with short-range coefficients $B^{(n)} \in L^1([x_0, \infty); (1 + |x|)dx)^{2m\times 2m}$, where iterations of Volterra-type integral equations yield the asymptotic high-energy expansion of $M_+(z, x_0)$ as $|z| \to \infty$ to any order, cf. Lemma 4.1.) This is not because of a lack of interest in this type of problem (we will discuss its relevance below), but simply since it is a nontrivial one, which, in many of its aspects, must be regarded as more difficult than the

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corresponding matrix-valued Schrödinger operator case, which in turn, was only very recently settled in [20]. The results proven in this paper show that in leading order (and independently of the self-adjoint boundary condition chosen at x_0),

$$M_+(z,x_0) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} iI_m + o(1), \tag{1.3}$$

where C_ε denotes the open sector in the open upper complex half-plane $\mathbb{C}_+$ with vertex at zero, symmetry axis along the positive imaginary axis, and opening angle ε, with $0 < \varepsilon < \pi/2$. We are interested in proving the asymptotic expansion (1.3) and especially in its higher-order analogs in powers of $1/z$, under optimal smoothness hypotheses on B. Such results are then also derived for the $2m \times 2m$ analog $M(z,x)$ of $M_+(z,x)$ associated with Dirac-type operators on $\mathbb{R}$.

Our principal motivation in studying this problem stems from our general interest in operator-valued Herglotz functions (cf. [17], [40], [41], [43], [44], [45], [46], [51], [113]) and their possible applications in the areas of inverse spectral theory and completely integrable systems. More precisely, using higher-order asymptotic expansions of $M_+(z,x)$, one can prove trace formulas for $B(x)$ and certain higher-order differential polynomials in $B(x)$ (similar in spirit to an approach pioneered in [48] (see also [37], [39]) in connection with Schrödinger operators). These trace formulas, in turn, then can be used to prove various results in inverse spectral theory for matrix-valued Dirac-type operators $D = J\frac{d}{dx} - B$ in $L^2(\mathbb{R})^{2m}$. For instance, using one of the principal results of this paper, Theorem 4.7, and its straightforward application to the asymptotic high-energy expansion of the diagonal Green's matrix $G(z,x,x) = (D-z)^{-1}(x,x)$ of D, the following matrix-valued analog of a classical uniqueness result of Borg [15] for one-dimensional Schrödinger operators will be proven in in the context of Dirac-type operators in Section 6.

Theorem 1.1. *Suppose that B is of the normal form $B(x) = \begin{pmatrix} B_{1,1}(x) & B_{1,2}(x) \\ B_{1,2}(x) & -B_{1,1}(x) \end{pmatrix}$, with $B_{1,1}(x)$ and $B_{1,2}(x)$ self-adjoint for a.e. $x \in \mathbb{R}$, and assume that D is reflectionless* (*e.g., B is periodic and D has uniform spectral multiplicity* $2m$). *In addition, suppose that D has spectrum equal to $\mathbb{R}$. Then,*

$$B(x) = 0 \textit{ for a.e. } x \in \mathbb{R}. \tag{1.4}$$

For related results see, for instance, [1], [2], [23], [36], [47], [53], [55]. Incidentally, the higher-order differential polynomials in $B(x)$ just alluded to represent the Ablowitz-Kaup-Newell-Segur (AKNS) or Zakharov-Shabat (ZS) invariants (i.e., densities associated with the AKNS-ZS conservation laws) and hence provide a link to infinite-dimensional completely integrable systems (cf., e.g., [7], [18], [27], [28], [29], [90], [94], [108], [111], [110], [112], and the references therein), especially, hierarchies of matrix-valued (i.e., nonabelian) nonlinear Schrödinger equations.

Although various aspects of inverse spectral theory for scalar Schrödinger, Jacobi, and Dirac-type operators, and more generally, for 2×2 Hamiltonian systems, are well-understood by now (cf. the extensive list of references provided in [41]), the corresponding theory for such operators and Hamiltonian systems with $m \times m$ matrix-valued coefficients, $m \in \mathbb{N}$, is still in its infancy. A particular inverse spectral theory aspect we have in mind is that of determining isospectral sets (manifolds) of such systems. It may, perhaps, come as a surprise that determining the isospectral set of Hamiltonian systems with matrix-valued periodic coefficients is a completely open problem. It appears to be no exaggeration to claim that absolutely

nothing seems to be known about the corresponding isospectral sets of periodic Dirac-type operators in the case $m \geq 2$. (More or less the same ignorance applies to Schrödinger, Jacobi, and more generally, to periodic $2m \times 2m$ Hamiltonian systems with $m \geq 2$.) Theorem 1.1 can be viewed as a first (and very modest) step toward the construction of isospectral manifolds of certain classes of matrix-valued potential coefficients B for Dirac-type operators.

However, asymptotic high-energy expansions for Weyl-Titchmarsh matrices on half-lines and on $\mathbb{R}$, their applications to trace formulas for $B(x)$, and the derivation of Borg-type theorems for Dirac operators are not the only topics under consideration in this paper. We also provide a comprehensive and new treatment of local uniqueness theorems for B in terms of exponentially close Weyl-Titchmarsh matrices. More precisely, in Section 5 we will prove the following result ($\|\cdot\|_{\mathbb{C}^{m\times m}}$ denotes a matrix norm on $\mathbb{C}^{m\times m}$).

Theorem 1.2. *Fix $x_0 \in \mathbb{R}$ and suppose that $B_j \in L^1([x_0, x_0+R])^{2m\times 2m}$ for all $R > 0$, posseses the normal form given in Theorem 1.1 a.e. on (x_0,∞), $j = 1,2$. Denote by $M_{j,+}(z,x)$, $x \geq x_0$, the unique Weyl-Titchmarsh matrices corresponding to the half-line Dirac-type operators in $L^2([x_0,\infty))^{2m}$ associated with B_j, $j = 1,2$ (fixing some self-adjoint boundary condition at x_0). Then,*

$$\text{if for some } a > 0,\ B_1(x) = B_2(x) \text{ for a.e. } x \in (x_0, x_0+a), \tag{1.5}$$

one obtains

$$\|M_{1,+}(z,x_0) - M_{2,+}(z,x_0)\|_{\mathbb{C}^{m\times m}} \underset{\substack{|z|\to\infty\\ z\in\rho_+}}{=} O\big(e^{-2\mathrm{Im}(z)a}\big) \tag{1.6}$$

along any ray $\rho_+ \subset \mathbb{C}_+$ with $0 < \arg(z) < \pi$ (and for all self-adjoint boundary condition at x_0). Conversely, if $m > 1$, assume in addition that $B_j \in L^\infty([x_0, x_0+a])^{2m\times 2m}$, $j = 1,2$. Moreover, suppose that for all $\varepsilon > 0$,

$$\|M_{1,+}(z,x_0) - M_{2,+}(z,x_0)\|_{\mathbb{C}^{m\times m}} \underset{\substack{|z|\to\infty\\ z\in\rho_{+,\ell}}}{=} O\big(e^{-2\mathrm{Im}(z)(a-\varepsilon)}\big), \quad \ell = 1,2, \tag{1.7}$$

along a ray $\rho_{+,1} \subset \mathbb{C}_+$ with $0 < \arg(z) < \pi/2$ and along a ray $\rho_{+,2} \subset \mathbb{C}_+$ with $\pi/2 < \arg(z) < \pi$. Then

$$B_1(x) = B_2(x) \text{ for a.e. } x \in [x_0, x_0+a]. \tag{1.8}$$

We also prove the analog of Theorem 1.2 for the $2m\times 2m$ Weyl-Titchmarsh matrices $M_j(z,x)$ associated with Dirac-type operators on $\mathbb{R}$ corresponding to B_j, $j = 1,2$.

In the context of scalar Schrödinger operators, the analog of Theorem 1.2 was first proved by Simon [114]. An alternative proof, applicable to matrix-valued Schrödinger operators was presented in [50] (cf. also [41]). More recently, yet another proof was found by Bennewitz [13] (following some ideas in [16]). In fact, our proof of Theorem 1.2 is based on that of Bennewitz [13] with additional modifications necessary to accomodate Dirac-type operators. These results extend the classical (global) uniqueness results due to Borg [16] and Marchenko [91], [92] which state that half-line m-functions uniquely determine the corresponding potential coefficient. The Dirac-type results such as Theorem 1.2 appear to be new, even in the special case $m = 1$. Previous results in the Dirac case focused on global uniqueness questions only. We refer to Gasymov and Levitan [34] in the case $m = 1$ and to Lesch and Malamud [81] in the matrix case $m \in \mathbb{N}$.

Next, we briefly sketch the content of each section. Section 2 provides the necessary background results on Dirac-type operators and recalls the basic notions of Weyl-Titchmarsh theory for Hamiltonian systems on a half-line as well as on $\mathbb{R}$, as developed in detail by Hinton and Shaw in a series of papers [62]–[66] (see also [8], [71], [67], [68], [73], [74], [78], [79], [80], [82], [99], [110]). In fact, most of these references deal with more general singular Hamiltonian systems and hence we specialize some of this material to the Dirac-type operator case at hand. While our treatment of Weyl-Titchmarsh theory in Section 2 is somewhat detailed, the results presented appear to be of vital importance for our asymptotic expansions in Sections 3 and 4. At any rate, we intended to present this material as concisely as possible.

Section 3 is devoted to a proof of the leading-order for the asymptotic high-energy expansion (1.3) of $M_+(z,x)$ for the Dirac case. We follow the strategy developed in the context of matrix-valued Schrödinger operators in our joint paper [20] by appealing to the theory of Riccati equations. By doing so, we follow the lead of Atkinson who highlighted the importance of Riccati equations, in this regard, first in [9], subsequently in [10], [11] and ultimately in the unpublished manuscript [12] in which he obtains the leading order for the asymptotic high-energy expansion of $M_+(z,x)$ for the matrix-valued Schrödinger case.

Theorems 3.4 and 3.6 contain two characterizations of the *Weyl disk* (cf. Definition 2.7). These characterizations provide an answer in Remark 3.7 to a point raised in [20] concerning the nature of the Weyl disk. From these characterizations of the Weyl disk, we obtain a realization of $M_+(z,x)$ as a differentiable function of x which satisfies a certain Riccati equation globally and whose imaginary part is strictly positive. We observe, in Remark 3.5, that the totality of Weyl disks, $D_+(z,x)$ (cf. Defintion 2.12), represents the phase space for these solutions. Thus, the asymptotic expansion we seek, represents the asymptotic high-energy behavior for certain solutions of a given Riccati equation.

Section 4 develops a systematic higher-order high-energy asymptotic expansion of $M_+(z,x)$ as $|z|\to\infty$, combining the leading-order asymptotic result in Section 3 with matrix-valued extensions of some methods based again on an associated Riccati equation. More precisely, following a technique in [49] in the scalar Schrödinger operator context, we show how to derive the general high-energy asymptotic expansion of $M_+(z,x)$ as $|z|\to\infty$ by combining Atkinson's leading-order term in (1.3) and the corresponding asymptotic expansion of $M_+(z,x)$ in the special case where B has compact support. Section 5 then contains our new local uniqueness results for $B(x)$ in terms of exponentially small differences of Weyl-Titchmarsh matrices as indicated in Theorem 1.2. Finally, in Section 6 we derive a new trace formula for Dirac-type operators D in $L^2(\mathbb{R})^{2m}$, using appropriate Herglotz representation results for the diagonal Green's matrix $G(z,x,x)$ discussed in Section 2. Moreover, we derive the Borg-type Theorem 1.1 for Dirac operators and close with an application to the case of periodic potentials coefficients B.

2. Weyl-Titchmarsh Matrices for Hamiltonian Systems

We now turn to the Weyl-Titchmarsh theory for Hamiltonian systems as developed by Hinton and Shaw in a series of papers devoted to the spectral theory of (singular) Hamiltonian systems [62]–[66] (see also [67], [68], [79], [80], [105], [110], [111], [112]). Throughout this paper all matrices will be considered over the field

of complex numbers $\mathbb{C}$. The basic assumptions throughout are described in the following three hypotheses.

Hypothesis 2.1. *Fix $m \in \mathbb{N}$ and define the $2m \times 2m$ matrix*

$$J = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix}, \tag{2.1a}$$

where I_m denotes the identity matrix in $\mathbb{C}^{m\times m}$. Suppose

$$A_{j,k}, B_{j,k} \in L^1_{\mathrm{loc}}(\mathbb{R})^{m\times m}, \quad j,k = 1,2 \tag{2.1b}$$

and assume

$$A(x) = \begin{pmatrix} A_{1,1}(x) & A_{1,2}(x) \\ A_{2,1}(x) & A_{2,2}(x) \end{pmatrix} \geq 0, \tag{2.1c}$$

$$B(x) = \begin{pmatrix} B_{1,1}(x) & B_{1,2}(x) \\ B_{2,1}(x) & B_{2,2}(x) \end{pmatrix} = B(x)^*, \tag{2.1d}$$

for a.e. $x \in \mathbb{R}$.

$L^1_{\mathrm{loc}}(\mathbb{R})$ denotes the set of locally integrable functions on $\mathbb{R}$. With $M \in \mathbb{C}^{m\times m}$, let M^t denote the transpose, let M^* denote the adjoint or conjugate transpose of the matrix M and let $M \geq 0$ and $M \leq 0$ denote nonnegative and nonpositive matrices M (i.e., positive and negative semidefinite matrices). Moreover, let $\mathrm{Im}(M) = (M - M^*)/(2i)$ and $\mathrm{Re}(M) = (M + M^*)/2$ denote, respectively, the imaginary and real parts of the matrix M.

Given Hypothesis 2.1, our Hamiltonian system is given by

$$J\Psi'(z,x) = (zA(x) + B(x))\Psi(z,x), \quad z \in \mathbb{C} \tag{2.2a}$$

for a.e. $x \in \mathbb{R}$, where z plays the role of the spectral parameter, and where

$$\Psi(z,x) = \begin{pmatrix} \psi_1(z,x) \\ \psi_2(z,x) \end{pmatrix}, \quad \psi_j(z,\cdot) \in AC_{\mathrm{loc}}(\mathbb{R})^{m\times r}, \ j = 1,2. \tag{2.2b}$$

$AC_{\mathrm{loc}}(\mathbb{R})$ denotes the set of locally absolutely continuous functions on $\mathbb{R}$. The parameter r in (2.2b) will be context dependent and range between $1 \leq r \leq m$.

For our discussions of the Weyl-Titchmarsh theory for the Hamiltonian system (2.2), we introduce the definiteness assumption found in Atkinson [8].

Hypothesis 2.2. *For all nontrivial solutions Ψ of* (2.2a) *with $r = 1$ in* (2.2b)*, we assume that*

$$\int_a^b dx\, \Psi(z,x)^* A(x)\Psi(z,x) > 0\ , \tag{2.3}$$

for every interval $(a,b) \subset \mathbb{R}$, $a < b$.

A principal example of such a system is the Dirac-type system obtained when

$$A(x) = I_{2m}, \tag{2.4}$$

and the subject of the present paper; another example being the matrix-valued Schrödinger system, obtained when

$$A(x) = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}, \qquad B(x) = \begin{pmatrix} -Q(x) & 0 \\ 0 & I_m \end{pmatrix}, \tag{2.5}$$

and the subject of [20]. When (2.5) holds, we note that (2.2a) is equivalent to

$$-\psi_1''(z,x) + Q(x)\psi_1(z,x) = z\psi_1(z,x), \tag{2.6}$$

$$\psi_2(z,x) = \psi_1'(z,x) \tag{2.7}$$

for a.e. $x \in \mathbb{R}$. Hypothesis 2.2 clearly holds in both examples.

Next, we introduce a set of matrices that will serve as boundary data for separated boundary conditions.

Hypothesis 2.3. *Let $\gamma = (\gamma_1\ \gamma_2)$ with $\gamma_j \in \mathbb{C}^{m\times m}$, $j = 1,2$. We assume that γ satisfies the following conditions,*

$$\operatorname{rank}(\gamma) = m, \tag{2.8a}$$

and that either

$$\operatorname{Im}(\gamma_2\gamma_1^*) \le 0, \quad \text{or} \quad \operatorname{Im}(\gamma_2\gamma_1^*) \ge 0, \tag{2.8b}$$

where $(2i)^{-1}\gamma J\gamma^* = \operatorname{Im}(\gamma_2\gamma_1^*)$. Given the rank condition in (2.8a)*, we assume, without loss of generality in what follows, the normalization*

$$\gamma\gamma^* = I_m. \tag{2.8c}$$

Remark 2.4. With $\alpha \in \mathbb{C}^{m\times 2m}$, the conditions

$$\alpha\alpha^* = I_m, \quad \alpha J\alpha^* = 0 \tag{2.9}$$

imply that α satisfies Hypothesis 2.3, and they explicitly read

$$\alpha_1\alpha_1^* + \alpha_2\alpha_2^* = I_m, \quad \alpha_2\alpha_1^* - \alpha_1\alpha_2^* = 0. \tag{2.10}$$

In fact, one also has

$$\alpha_1^*\alpha_1 + \alpha_2^*\alpha_2 = I_m, \quad \alpha_2^*\alpha_1 - \alpha_1^*\alpha_2 = 0, \tag{2.11}$$

as is clear from

$$\begin{pmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_1 \end{pmatrix} \begin{pmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{pmatrix} = I_{2m} = \begin{pmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 \\ -\alpha_2 & \alpha_1 \end{pmatrix}, \tag{2.12}$$

since any left inverse matrix is also a right inverse, and vice versa. Moreover, from (2.11) we obtain

$$\alpha^*\alpha J + J\alpha^*\alpha = J. \tag{2.13}$$

With $\alpha \in \mathbb{C}^{m\times 2m}$ satisfying (2.9), let $\Psi(z,x,x_0,\alpha)$ be a normalized fundamental system of solutions of (2.2) at some $x_0 \in \mathbb{R}$. That is, $\Psi(z,x,x_0,\alpha)$ satisfies (2.2) for a.e. $x \in \mathbb{R}$, and

$$\Psi(z,x_0,x_0,\alpha) = (\alpha^*\ J\alpha^*) = \begin{pmatrix} \alpha_1^* & -\alpha_2^* \\ \alpha_2^* & \alpha_1^* \end{pmatrix}. \tag{2.14a}$$

We partition $\Psi(z,x,x_0,\alpha)$ as follows,

$$\begin{aligned} \Psi(z,x,x_0,\alpha) &= (\Theta(z,x,x_0,\alpha)\ \Phi(z,x,x_0,\alpha)) && (2.14\text{b}) \\ &= \begin{pmatrix} \theta_1(z,x,x_0,\alpha) & \phi_1(z,x,x_0,\alpha) \\ \theta_2(z,x,x_0,\alpha) & \phi_2(z,x,x_0,\alpha) \end{pmatrix}, && (2.14\text{c}) \end{aligned}$$

where $\theta_j(z,x,x_0,\alpha)$ and $\phi_j(z,x,x_0,\alpha)$ for $j = 1,2$ are $m\times m$ matrices, entire with respect to $z \in \mathbb{C}$, and normalized according to (2.14a). One can now prove the following result.

Lemma 2.5. *Let $\Theta(z,x,x_0,\alpha)$ and $\Phi(z,x,x_0,\alpha)$ be defined in* (2.14b) *with α and β satisfying Hypothesis 2.3 and with $\mathrm{Im}(\alpha_2\alpha_1^*)=0$. Then, for $c\neq x_0$, $\beta\Phi(z,c,x_0,\alpha)$ is singular if and only if z is an eigenvalue for the regular boundary value problem given by* (2.2a) *on $[x_0,c]$ if $c>x_0$ and on $[c,x_0]$ if $c<x_0$ together with the separated boundary conditions*

$$\alpha\Psi(z,x_0)=0, \quad \beta\Psi(z,c)=0, \tag{2.15}$$

where $\Psi(z,x)=(\psi_1(z,x)^t\ \psi_2(z,x)^t)^t$ with $\psi_j(z,\cdot)\in AC([x_0,c])$ if $c>x_0$ and $\psi_j(z,\cdot)\in AC([c,x_0])$ if $c<x_0$, $j=1,2$.

Note that the regular boundary value problem described in Lemma 2.5 is self-adjoint when $\mathrm{Im}(\beta_2\beta_1^*)=0$.

In light of Lemma 2.5, it is possible to introduce, under appropriate conditions, the $m\times m$ matrix-valued function, $M(z,c,x_0,\alpha,\beta)$, as follows.

Definition 2.6. Let $\Theta(z,x,x_0,\alpha)$, and $\Phi(z,x,x_0,\alpha)$ be defined in (2.14b) with α and β satisfying Hypothesis 2.3 and with $\mathrm{Im}(\alpha_2\alpha_1^*)=0$. For $c\neq x_0$, and $\beta\Phi(z,c,x_0,\alpha)$ nonsingular let

$$M(z,c,x_0,\alpha,\beta)=-[\beta\Phi(z,c,x_0,\alpha)]^{-1}[\beta\Theta(z,c,x_0,\alpha)]. \tag{2.16}$$

$M(z,c,x_0,\alpha,\beta)$ is said to be the *Weyl-Titchmarsh M-function* for the regular boundary value problem described in Lemma 2.5.

The Weyl-Titchmarsh M-function is an $m\times m$ matrix-valued function with meromorphic entries whose poles correspond to eigenvalues for the regular boundary value problem given by (2.2a) and (2.15). Moreover, if $M\in\mathbb{C}^{m\times m}$, and one defines

$$U(z,x,x_0,\alpha)=\begin{pmatrix}u_1(z,x,x_0,\alpha)\\ u_2(z,x,x_0,\alpha)\end{pmatrix}=\Psi(z,x,x_0,\alpha)\begin{pmatrix}I_m\\ M\end{pmatrix}, \tag{2.17}$$

with $u_j(z,x,x_0,\alpha)\in\mathbb{C}^{m\times m}$, $j=1,2$, then $U(z,x,x_0,\alpha)$ will satisfy the boundary condition at $x=c$ in (2.15) whenever $M=M(z,c,x_0,\alpha,\beta)$. Intimately connected with the matrices introduced in Definition 2.6 is the set of $m\times m$ complex matrices known as the Weyl disk. Several characterization of this set have appeared in the literature (see, e.g., [8], [11], [12], [67], [62], [79], [99]). We now mention two, and will introduce two others in Section 3 which we use in the derivation of the asymptotic expansions that are the subject of Sections 3 and 4.

To describe this set, we first introduce the matrix-valued function $E_c(M)$: With $c\neq x_0$, $z\in\mathbb{C}\backslash\mathbb{R}$, and with $U(z,c,x_0,\alpha)$ defined by (2.17) in terms of a matrix $M\in\mathbb{C}^{m\times m}$, let

$$E_c(M)=\sigma(x_0,c,z)U(z,c,x_0,\alpha)^*(iJ)U(z,c,x_0,\alpha), \tag{2.18}$$

where

$$\sigma(s,t,z)=\frac{(s-t)\mathrm{Im}(z)}{|(s-t)\mathrm{Im}(z)|}, \quad \sigma(s,t)=\sigma(s,t,i), \quad \sigma(z)=\sigma(1,0,z), \tag{2.19}$$

with $s\neq t$, and $s,t\in\mathbb{R}$.

Definition 2.7. Let the following be fixed: Real numbers x_0 and $c\neq x_0$, an $m\times 2m$ matrix α satisfying (2.9), and $z\in\mathbb{C}\backslash\mathbb{R}$. $\mathcal{D}(z,c,x_0,\alpha)$ will denote the collection of all $M\in\mathbb{C}^{m\times m}$ for which $E_c(M)\leq 0$, where $E_c(M)$ is defined in (2.18). $\mathcal{D}(z,c,x_0,\alpha)$ is said to be a *Weyl disk*. The set of $M\in\mathbb{C}^{m\times m}$ for which $E_c(M)=0$ is said to be a *Weyl circle* (even when $m>1$).

This definition leads to a presentation that is a generalization of the description first given by Weyl [119]; a presentation which is geometric in nature, involves the contractive matrices $V \in \mathbb{C}^{m\times m}$, such that $VV^* \le I_m$, and provides the justification for the geometric terms of circle and disk (cf., e.g., [62], [67], [79], [99]).

The disk has also been characterized in terms of matrices which statisfy Hypothesis 2.3 and which serve as boundary data for the regular boundary value problem described in Lemma 2.5 (cf., e.g., [11], [12]). More precisely, one could have used the following alternative definition.

Definition 2.7A. $\mathcal{D}(z,c,x_0,\alpha)$ denotes the collection of all $M \in \mathbb{C}^{m\times m}$ obtained by the construction given in (2.16) where $c \neq x_0$, $z \in \mathbb{C}/\mathbb{R}$, where α and β are the $m\times m$ matrices defined in Hypothesis 2.3 for which $\sigma(c,x_0,z)\mathrm{Im}(\beta_2\beta_1^*) \ge 0$, and $\mathrm{Im}(\alpha_2\alpha_1^*) = 0$.

However, in this paper we take Definition 2.7 as our point of departure.

We note that the Weyl circle corresponds to the regular boundary value problems in Lemma 2.5 with separated, self-adjoint boundary conditions. For convenience of the reader, and to achieve a reasonable level of completeness, we reproduce the corresponding short proof below.

Lemma 2.8 ([65], [67], [79])**.** *Let $M \in \mathbb{C}^{m\times m}$, $c \neq x_0$, and $z \in \mathbb{C}\backslash\mathbb{R}$. Then, $E_c(M) = 0$ if and only if there is a $\beta \in \mathbb{C}^{m\times 2m}$ satisfying* (2.9) *such that*

$$0 = \beta U(z,c,x_0,\alpha), \tag{2.20}$$

where $U(z,c) = U(z,c,x_0,\alpha)$ is defined in (2.17) *in terms of M. With β so defined,*

$$M = -[\beta\Phi(z,c,x_0,\alpha)]^{-1}[\beta\Theta(z,c,x_0,\alpha)], \tag{2.21}$$

that is, $M = M(z,c,x_0,\alpha,\beta)$. Moreover, β may be chosen to satisfy (2.8c)*, and hence Hypothesis 2.3.*

Proof. Let $z \in \mathbb{C}\backslash\mathbb{R}$, and suppose for a given $M \in \mathbb{C}^{m\times m}$ that there is a $\beta \in \mathbb{C}^{m\times 2m}$ which satisfies (2.9) and such that (2.20) is satisfied. Given that $\beta J\beta^* = 2i\mathrm{Im}(\beta_2\beta_1^*) = 0$, and given that $\mathrm{rank}(\beta) = \mathrm{rank}(J\beta^*) = m$, there is a nonsingular $C \in \mathbb{C}^{m\times m}$ such that $U(z,c) = J\beta^* C$. Hence, $E_c(M) = i\sigma(c,x_0,z)C^*\beta J\beta^* C = 0$.

Upon showing that $\beta\Phi(z,c) = \beta\Phi(z,c,x_0,\alpha)$ is nonsingular, (2.21) will then follow from (2.20). If $\beta\Phi(z,c)$ is singular, then there are nonzero vectors $v,w \in \mathbb{C}^m$ such that $\beta\Phi(z,c)v = 0$, and such that $\Phi(z,c)v = J\beta^* w$. Let $\Psi_j = \Psi_j(z,x)$, $j=1,2$ denote solutions of (2.2a) with $z = z_j$, $j=1,2$. Then,

$$(\Psi_1^* J\Psi_2)' = (z_2 - \bar z_1)\Psi_1^* A\Psi_2. \tag{2.22}$$

Using (2.22), and recalling that $\Phi(z,x)$ is defined in (2.14), we obtain

$$2i\mathrm{Im}(z)\int_{x_0}^{c} dx\, v^*\Phi(z,x)^* A(x)\Phi(z,x)v = v^*\Phi(z,c)^* J\Phi(z,c)v \tag{2.23a}$$

$$= w^*\beta J\beta^* w = 0. \tag{2.23b}$$

Thus, by Hypothesis 2.2, $\mathrm{Im}(z) = 0$. This contradicts the assumption that $z \in \mathbb{C}\backslash\mathbb{R}$.

Conversely, if $E_c(M) = 0$ for a given $M \in \mathbb{C}^{m\times m}$, then for $z \in \mathbb{C}\backslash\mathbb{R}$ let $\beta = (I_m\ M^*)\Psi(z,c,x_0,\alpha)^* J = U(z,c,x_0,\alpha)^* J$. One observes that (2.20) is satisfied and that $\mathrm{rank}(\beta) = m$. Moreover, $0 = \sigma(x_0,c,z)E_c(M)/2 = \mathrm{Im}(\beta_2\beta_1^*)$. If for this choice of β, (2.8c) is not yet satisfied, one introduces $\Delta = (\beta\beta^*)^{-1/2}\beta$ and observes that $0 = \Delta U(z,c,x_0,\alpha)$, $\mathrm{Im}(\Delta_2\Delta_1^*) = (\beta\beta^*)^{-1/2}\mathrm{Im}(\beta_2\beta_1^*)(\beta\beta^*)^{-1/2}$, and that Δ satisfies all requirements of (2.9). $\square$

Next, we recall a fundamental property associated with matrices in $\mathcal{D}(z, c, x_0, \alpha)$.

Lemma 2.9. *If $M \in \mathcal{D}(z, c, x_0, \alpha)$, then*

$$\sigma(c, x_0, z)\mathrm{Im}(M) > 0. \tag{2.24}$$

Moreover, whenever $\beta \in \mathbb{C}^{m\times 2m}$ satisfies (2.9)*,*

$$M(\bar{z}, c, x_0, \alpha, \beta) = M(z, c, x_0, \alpha, \beta)^*. \tag{2.25}$$

Proof. Let $\Psi_j = \Psi_j(z, x)$, $j = 1, 2$ denote solutions of (2.2a) with $z = z_j$, $j = 1, 2$. Then $(\Psi_1^* J\Psi_2)' = (z_2 - \bar{z}_1)\Psi_1^* A\Psi_2$ as in (2.22). This implies

$$\begin{aligned} 2i\mathrm{Im}(z) \int_{x_0}^{c} dx\, U(z, x)^* A(x) U(z, x) &= U(z, x)^* J U(z, x)\big|_{x_0}^{c} \\ &= 2i\mathrm{Im}(M) + U(z, c)^* J U(z, c), \end{aligned} \tag{2.26}$$

with $U(z, x) = U(z, x, x_0, \alpha)$ defined in (2.17). Moreover, by the definition of $E_c(M)$ given in (2.18), one obtains

$$\begin{aligned} &2\sigma(c, x_0, z)\mathrm{Im}(M) \\ &= -E_c(M) + 2\sigma(c, x_0)|\mathrm{Im}(z)| \int_{x_0}^{c} ds\, U(z, s)^* A(s) U(z, s). \end{aligned} \tag{2.27}$$

By Hypothesis 2.2 and Definition 2.7, one infers that $\sigma(c, x_0, z)\mathrm{Im}(M) > 0$. To prove (2.25), let $\Psi(z, x) = \Psi(z, x, x_0, \alpha)$, where Ψ is defined in (2.2). Then, by (2.22),

$$\Psi(\bar{z}, x)^* J\Psi(z, x) = J, \tag{2.28}$$

which implies $J\Psi(z.x)(\Psi(\bar{z}, x)J)^* = I_{2m}$ and hence

$$\Psi(z, x)J\Psi(\bar{z}, x)^* = J. \tag{2.29}$$

Thus one concludes

$$\beta\Phi(z, c)(\beta\Theta(\bar{z}, c))^* - \beta\Theta(z, c)(\beta\Phi(\bar{z}, c))^* = \beta J\beta^* = 0, \tag{2.30}$$

from which (2.25) follows immediately by Lemma 2.8. □

For $c > x_0$, the function $M(z, c, x_0, \alpha, \beta)$, defined by (2.16), and satisfying (2.24), is said to be a matrix-valued *Herglotz* function of rank m. Hence, for $\mathrm{Im}(\beta_2\beta_1^*) = 0$, poles of $M(z, c, x_0, \alpha, \beta)$, $c > x_0$, are at most of first order, are real, and have nonpositive residues. Such functions admit a representation of the form

$$\begin{aligned} M(z, c, x_0, \alpha, \beta) &= C_1(c, x_0, \alpha, \beta) + zC_2(c, x_0, \alpha, \beta) \\ &\quad + \int_{-\infty}^{\infty} d\Omega(\lambda, c, x_0, \alpha, \beta) \left(\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2}\right), \quad c > x_0, \end{aligned} \tag{2.31}$$

where $C_2(c, x_0, \alpha, \beta) \geq 0$ and $C_1(c, x_0, \alpha, \beta)$ are self-adjoint $m \times m$ matrices, and where $\Omega(\lambda, c, x_0, \alpha, \beta)$ is a nondecreasing $m \times m$ matrix-valued function such that

$$\int_{-\infty}^{\infty} \|d\Omega(\lambda, c, x_0, \alpha, \beta)\|_{\mathbb{C}^{m\times m}} (1 + \lambda^2)^{-1} < \infty, \tag{2.32a}$$

$$\Omega((\lambda, \mu], c, x_0, \alpha, \beta) = \lim_{\delta\downarrow 0}\lim_{\epsilon\downarrow 0} \frac{1}{\pi} \int_{\lambda+\delta}^{\mu+\delta} d\nu\, \sigma(c, x_0)\mathrm{Im}\left(M(\nu + i\epsilon, c, x_0, \alpha, \beta)\right). \tag{2.32b}$$

In general, for self-adjoint boundary value problems, $\Omega(\lambda, c, x_0, \alpha, \beta)$ is piecewise constant with jump discontinuities precisely at the eigenvalues of the boundary value problem, and that in the matrix-valued Schrödinger and Dirac-type cases $C_2 = 0$ in (2.31) (and later in (2.63) and (2.78)). Analogous statements apply to $-M(z, c, x_0, \alpha, \beta)$ if $c < x_0$. For such problems, we note in the subsequent lemma that for fixed β, varying the boundary data α produces Weyl-Titchmarsh matrices $M(z, c, x_0, \alpha, \beta)$ related to each other via linear fractional transformations (see also [46], [51] for a general approach to such linear fractional transformations).

Lemma 2.10. *Suppose $\alpha, \beta, \gamma \in \mathbb{C}^{m\times 2m}$ satisfy* (2.9). *Let $M_\alpha = M(z, c, x_0, \alpha, \beta)$, and $M_\gamma = M(z, c, x_0, \gamma, \beta)$. Then,*

$$M_\alpha = [-\alpha J\gamma^* + \alpha\gamma^* M_\gamma][\alpha\gamma^* + \alpha J\gamma^* M_\gamma]^{-1}. \tag{2.33}$$

Proof. Let $U_\alpha(z, x) = U(z, x, x_0, \alpha)$ and $U_\gamma(z, x) = U(z, x, x_0, \gamma)$ be defined in (2.17) with $M = M_\alpha$ and $M = M_\gamma$ respectively. Then,

$$0 = \beta U_\alpha(z, c) = \gamma U_\gamma(z, c). \tag{2.34}$$

By the rank condition (2.8a),

$$U_\alpha(z, c) = J\beta^* C_\alpha\,, \qquad U_\gamma(z, c) = J\beta^* C_\gamma \tag{2.35}$$

for nonsingular $C_\alpha,\ C_\gamma \in \mathbb{C}^{m\times m}$. Thus, by (2.14a), and by the uniqueness of solution of (2.2a), there is a nonsingular $C \in \mathbb{C}^{m\times m}$ for which

$$\begin{pmatrix}\alpha^* & J\alpha^*\end{pmatrix}\begin{pmatrix} I_m \\ M_\alpha\end{pmatrix} = U_\alpha(z, x_0) = U_\gamma(z, x_0)C = \begin{pmatrix}\gamma^* & J\gamma^*\end{pmatrix}\begin{pmatrix} I_m \\ M_\gamma\end{pmatrix} C. \tag{2.36}$$

By (2.13),

$$\begin{pmatrix}\alpha^* & J\alpha^*\end{pmatrix}^{-1} = \begin{pmatrix}\alpha \\ -\alpha J\end{pmatrix}; \tag{2.37}$$

and hence, by (2.36) we see that

$$I_m = (\alpha\gamma^* + \alpha J\gamma^* M_\gamma)C \tag{2.38a}$$
$$M_\alpha = (-\alpha J\gamma^* + \alpha\gamma^* M_\gamma)C, \tag{2.38b}$$

from which (2.33) immediately follows. □

Remark 2.11. From the proof of the previous lemma one infers, in general, that

$$U_\gamma(z, x) = U_\alpha(z, x)(\alpha\gamma^* + \alpha J\gamma^* M_\gamma). \tag{2.39}$$

Moreover, if $\alpha_0 = (I_m\ 0)$ and $\gamma_0 = (0\ I_m)$ one observes, in particular,

$$M(z, c, x_0, \alpha_0, \beta) = -M(z, c, x_0, \gamma_0, \beta)^{-1}. \tag{2.40}$$

We further note that the sets $\mathcal{D}(z, c, x_0, \alpha)$ are closed, and convex, (cf., e.g., [65], [67], [79], [99]). Moreover, by (2.27) and Hypothesis 2.2, one concludes that $E_c(M)$ is strictly increasing. This fact together with Lemma 2.8 implies that, as a function of c, the sets $\mathcal{D}(z, c, x_0, \alpha)$ are strictly nesting in the sense that

$$\mathcal{D}(z, c_2, x_0, \alpha) \subset \mathcal{D}(z, c_1, x_0, \alpha) \quad \text{for} \quad x_0 < c_1 < c_2 \quad \text{or} \quad c_2 < c_1 < x_0. \tag{2.41}$$

Hence, the intersection of this nested sequence, as $c \to \pm\infty$, is nonempty, closed and convex. We say that this intersection is a limiting set for the nested sequence.

Definition 2.12. Let $\mathcal{D}_\pm(z,x_0,\alpha)$ denote the closed, convex set in the space of $m\times m$ matrices which is the limit, as $c\to\pm\infty$, of the nested collection of sets $\mathcal{D}(z,c,x_0,\alpha)$ given in Definition 2.7. $\mathcal{D}_\pm(z,x_0,\alpha)$ is said to be a limiting *disk*. Elements of $\mathcal{D}_\pm(z,x_0,\alpha)$ are denoted by $M_\pm(z,x_0,\alpha)\in\mathbb{C}^{m\times m}$.

In light of the containment described in (2.41), for $c\neq x_0$ and $z\in\mathbb{C}\backslash\mathbb{R}$,

$$\mathcal{D}_\pm(z,x_0,\alpha)\subset\mathcal{D}(z,c,x_0,\alpha), \tag{2.42}$$

with emphasis on strict containment of the disks in (2.42). Moreover, by (2.27),

$$M\in\mathcal{D}_\pm(z,x_0,\alpha) \text{ precisely when } E_c(M)<0 \text{ for all } c\in(x_0,\pm\infty). \tag{2.43}$$

The following Lemma appears to have gone unnoted in the literature.

Lemma 2.13. *Let $M\in\mathbb{C}^{m\times m}$, $c\neq x_0$, and $z\in\mathbb{C}\backslash\mathbb{R}$. Then, $E_c(M)<0$ if and only if there is a $\beta\in\mathbb{C}^{m\times 2m}$ satisfying the condition*

$$\sigma(c,x_0,z)\operatorname{Im}(\beta_2\beta_1^*)>0, \tag{2.44}$$

and such that (2.20) *holds with $u_j(z,c)=u_j(z,c,x_0,\alpha)$, $j=1,2$, defined in* (2.17) *in terms of M. With β so defined,* (2.21) *holds; that is, $M=M(z,c,x_0,\alpha,\beta)$. Moreover, β maybe chosen to satisfy* (2.8c)*, and hence Hypothesis 2.3.*

Proof. Let $z\in\mathbb{C}\backslash\mathbb{R}$, and for a given $M\in\mathbb{C}^{m\times m}$ suppose that there is a $\beta\in\mathbb{C}^{m\times 2m}$ satisfying (2.44) such that (2.20) holds. The matrices β_j, $j=1,2$, are invertible by (2.44), and by (2.20) it follows that

$$U(z,c)=\begin{pmatrix}-\beta_1^{-1}\beta_2\\ I_m\end{pmatrix}u_2(z,c). \tag{2.45}$$

By (2.18) and (2.45), one then concludes that

$$E_c(M)=-2\sigma(c,x_0,z)u_2(z,c)^*\beta_1^{-1}\operatorname{Im}(\beta_2\beta_1^*)(\beta_1^*)^{-1}u_2(z,c), \tag{2.46}$$

and hence that $E_c(M)<0$ whenever (2.44) holds.

Upon showing that $\beta\Phi(z,c)$ is nonsingular, (2.21) will follow from (2.20). If $\beta\Phi(z,c)$ is singular, then there is a nonzero vector $v\in\mathbb{C}^m$ such that $\beta\Phi(z,c)v=0$. By the nonsingularity of β_j, $j=1,2$, $\phi_1(z,c)v=-\beta_1^{-1}\beta_2\phi_2(z,c)v$, and as a result, (2.23a) yields

$$\begin{aligned}&2\sigma(c,x_0)|\operatorname{Im}(z)|\int_{x_0}^{c}dx\,v^*\Phi(z,x)^*A(x)\Phi(z,x)v\\ &\quad=-2\sigma(c,x_0,z)v^*\phi_2(z,c)^*\beta_1^{-1}\operatorname{Im}(\beta_2\beta_1^*)(\beta_1^*)^{-1}\phi_2(z,c)v,\end{aligned} \tag{2.47}$$

and hence, a contradiction given (2.44) (cf. (2.3)).

Conversely, if $E_c(M)<0$ for a given $M\in\mathbb{C}^{m\times m}$, then for $z\in\mathbb{C}\backslash\mathbb{R}$, $u_j(z,c)$, $j=1,2$, defined by (2.17), are nonsingular. Indeed, if either $u_1(z,c)$ or $u_2(z,c)$ are singular, then there is a $v\in\mathbb{C}^m$, $v\neq0$, such that $v^*E_c(M)v=0$, a contradiction. Next, let $\beta_1=I_m$ and let $\beta_2=-u_1(z,c)u_2(z,c)^{-1}$. Then, for these β_j, $j=1,2$, (2.20) holds. Equation (2.46) now implies that $\sigma(c,x_0,z)\operatorname{Im}(\beta_2\beta_1^*)>0$ for $c\neq x_0$ and $z\in\mathbb{C}\backslash\mathbb{R}$. For this choice, β does not satisfy (2.8c). However, one can normalize β as described in the proof of Lemma 2.8. ∎

Hence by Lemma 2.13 and (2.43), we see that if $M\in\mathcal{D}_\pm(z,x_0,\alpha)$, then for some $\beta\in\mathbb{C}^{m\times 2m}$ satisfying (2.44)

$$M_\pm(z,x_0,\alpha)=M(z,c,x_0,\alpha,\beta). \tag{2.48}$$

Remark 2.14. To the reader of [20], our study of the high-energy asymptotics of the Weyl-Titchmarsh M-function for matrix-Schrödinger operators, we offer this cautionary note: In [20], $D(z, c, x_0, \alpha)$ represents the set of matrices characterized by Lemmas 2.8 and 2.13. However, the homeomorphism that exists between the contractive matrices $V \in \mathbb{C}^{m\times m}$, $VV^* \leq I_m$, and the Weyl disk, $D(z, c, x_0, \alpha)$, (cf., [65], [67], [79], [99]) shows that those $M \in \mathbb{C}^{m\times m}$ characterized in Lemma 2.8 correspond to the set of unitary matrices while those characterized in Lemma 2.13 correspond to the contractive matrices for which $VV^* < I_m$. Hence, Lemma 2.8 characterizes part of the boundary while Lemma 2.13 characterizes the interior of the Weyl disk as it is defined in Defintion 2.7. As a result, the closure of the set consisting of those $M \in \mathbb{C}^{m\times m}$ characterized by these two lemmas (i.e., those M which correspond to $VV^* < I_m$, or to $VV^* = I_m$) is the Weyl disk. Thus, for deriving high-energy asymptotics for $M_\pm(z, x_0, \alpha)$, it is sufficient to consider the subset of the Weyl disk consisting of those matrices, $M \in \mathbb{C}^{m\times m}$, characterized in Lemma 2.8 and Lemma 2.13. This was the approach taken in [20].

When $\mathcal{D}_\pm(z, x_0, \alpha)$ is a singleton matrix, the system (2.2a) is said to be in the *limit point* (l.p.) case at $\pm\infty$. If $\mathcal{D}_\pm(z, x_0, \alpha)$ has nonempty interior, then (2.2a) is said to be in the *limit circle* (l.c.) case at $\pm\infty$. Indeed, for the case $m = 1$, the limit point case corresponds to a point in $\mathbb{C}$, whereas the limit circle case corresponds to $\mathcal{D}_\pm(z, x_0, \alpha)$ being a disk in $\mathbb{C}$.

These apparent geometric properties for the disk correspond to analytic properties for the solutions of the Hamiltonian system (2.2a). To recall this correspondence, we introduce the following spaces in which we assume that $-\infty \leq a < b \leq \infty$,

$$L^2_A((a, b)) = \left\{\phi : (a, b) \to \mathbb{C}^{2m} \,\middle|\, \int_a^b dx\, (\phi(x), A\phi(x))_{\mathbb{C}^{2m}} < \infty\right\}, \tag{2.49a}$$

$$N(z, \infty) = \{\phi \in L^2_A((c, \infty)) \mid J\phi' = (zA + B)\phi \text{ a.e. on } (c, \infty)\}, \tag{2.49b}$$

$$N(z, -\infty) = \{\phi \in L^2_A((-\infty, c)) \mid J\phi' = (zA + B)\phi \text{ a.e. on } (-\infty, c)\}, \tag{2.49c}$$

for some $c \in \mathbb{R}$ and $z \in \mathbb{C}$. (Here $(\phi, \psi)_{\mathbb{C}^n} = \sum_{j=1}^n \overline{\phi}_j \psi_j$ denotes the standard scalar product in $\mathbb{C}^n$, abbreviating $\chi \in \mathbb{C}^n$ by $\chi = (\chi_1, \dots, \chi_n)^t$.) Both dimensions of the spaces in (2.49b) and (2.49c), $\dim_{\mathbb{C}}(N(z, \infty))$ and $\dim_{\mathbb{C}}(N(z, -\infty))$, are constant for $z \in \mathbb{C}_\pm = \{\zeta \in \mathbb{C} \mid \pm\mathrm{Im}(\zeta) > 0\}$ (see, e.g., [8], [74]). One then observes that the Hamiltonian system (2.2a) is in the limit point case at $\pm\infty$ whenever

$$\dim_{\mathbb{C}}(N(z, \pm\infty)) = m \text{ for all } z \in \mathbb{C}\backslash\mathbb{R} \tag{2.50}$$

and in the limit circle case at $\pm\infty$ whenever

$$\dim_{\mathbb{C}}(N(z, \pm\infty)) = 2m \text{ for all } z \in \mathbb{C}. \tag{2.51}$$

Next we show that the Dirac-type systems considered in this paper are always in the limit point case at $\pm\infty$. Results of this type, under varying sets of assumptions on $B(x)$, are well-known to experts in the field. For instance, in the case $m = 1$ and with $B_{1,2}(x) = B_{2,1}(x)$ this fact can be found in [118]. For $B \in C(\mathbb{R})^{2m\times 2m}$ and a more general constant matrix A, this result is proven in [81] (their proof, however, extends to the current $B \in L^1_{\mathrm{loc}}(\mathbb{R})$ case). More generally, multi-dimensional Dirac operators with $L^2_{\mathrm{loc}}(\mathbb{R}^n)$-type coefficients (and additional conditions) can be found in [83]. A short proof in the case $m = 1$ has recently been sent to us by Don Hinton [58]. For convenience of the reader we present its elementary generalization to $m \in$

$\mathbb{N}$ below (see also [19] for a sketch of such a proof). After completion of this paper we became aware of a recent preprint by Lesch and Malamud [82] which provides a thorough study of self-adjointness questions for more general Hamiltonian systems than those studied in this paper.

Lemma 2.15. *The limit point case holds for Dirac-type systems* (*i.e., for* $A = I_{2m}$ *in* (2.2a)) *at* $\pm\infty$.

Proof. Let $\{y_\ell(z,x)\}_{\ell=1,\dots,k}$ and $\{w_j(z,x)\}_{j=1,\dots,k'}$ denote bases for $N(z,\pm\infty)$ and $N(\overline{z},\pm\infty)$, respectively. By Theorem 9.11.1 of Atkinson [8], one has $k,k' \geq m$ for $z \in \mathbb{C}\backslash\mathbb{R}$. We now assume that $k > m$.

One observes that $\{y_1(z,c),\dots,y_k(z,c)\}$ and $w_1(\overline{z},c),\dots,w_{k'}(\overline{z},c)\}$ are linearly independent in $\mathbb{C}^{2m+1}$, where $k+k' \geq 2m+1$. Consequently, there is some $s \in \{1,\dots,k\}$ and some $r \in \{1,\dots,k'\}$ such that

$$w_r(\overline{z},c)^* J y_s(z,c) \neq 0. \tag{2.52}$$

By Lagrange's identity,

$$w_r(\overline{z},x)^* J y_s(z,x) = w_r(\overline{z},c)^* J y_s(z,c) \tag{2.53}$$

is constant with respect to x. On the other hand, an application of Cauchy's inequality shows that the left-hand side of (2.53) is in $L^1((c,\pm\infty))$. By (2.52) one obtains a contradiction and hence concludes that

$$\dim_{\mathbb{C}}(N(z,\pm\infty)) = m. \tag{2.54}$$

The analogous argument then also yields

$$\dim_{\mathbb{C}}(N(\overline{z},\pm\infty)) = m \tag{2.55}$$

and hence the limit point property of Dirac-type systems with $A(x) = I_{2m}$ in (2.2a). $\square$

Returning to the general case (2.2a), in either the limit point or limit circle cases, $M_\pm(z,x_0,\alpha) \in \partial\mathcal{D}_\pm(z,x_0,\alpha)$ is said to be a *half-line Weyl-Titchmarsh matrix.* Each such matrix is associated with the construction of a self-adjoint operator acting on $L^2_A([x_0,\pm\infty)) \cap \mathrm{AC}([x_0,\pm\infty))^{2m}$ for the Hamiltonian system (2.2a). However, for those intermediate cases where $m < \dim_{\mathbb{C}}(N(z,\pm\infty)) < 2m$, Hinton and Schneider have noted that not every element of $\partial\mathcal{D}_\pm(z,x_0,\alpha)$ is a half-line Weyl-Titchmarsh matrix, and have characterized those elements of the boundary that are (cf. [67], [68]).

For convenience of the reader we summarize some of the principal results on half-line Weyl-Titchmarsh matrices next.

Theorem 2.16 ([3], [17], [51], [62], [63], [66], [78])**.** *Suppose Hypotheses 2.1 and 2.2. Let* $z \in \mathbb{C}\backslash\mathbb{R}$*,* $x_0 \in \mathbb{R}$*, and denote by* $\alpha,\gamma \in \mathbb{C}^{m\times 2m}$ *matrices satisfying* (2.9)*. Then,*
(*i*) $\pm M_\pm(z,x_0,\alpha)$ *is an* $m\times m$ *matrix-valued Herglotz function of maximal rank. In particular,*

$$\mathrm{Im}(\pm M_\pm(z,x_0,\alpha)) > 0, \quad z \in \mathbb{C}_+, \tag{2.56}$$

$$M_\pm(\overline{z},x_0,\alpha) = M_\pm(z,x_0,\alpha)^*, \tag{2.57}$$

$$\mathrm{rank}(M_\pm(z,x_0,\alpha)) = m, \tag{2.58}$$

$$\lim_{\varepsilon\downarrow 0} M_\pm(\lambda+i\varepsilon,x_0,\alpha) \text{ exists for a.e. } \lambda \in \mathbb{R}, \tag{2.59}$$

$$\begin{aligned} M_\pm(z,x_0,\alpha) = [-\alpha J\gamma^* + \alpha\gamma^* M_\pm(z,x_0,\gamma)] \times \\ \times [\alpha\gamma^* + \alpha J\gamma^* M_\pm(z,x_0,\gamma)]^{-1}. \end{aligned} \tag{2.60}$$

Local singularities of $\pm M_\pm(z,x_0,\alpha)$ *and* $\mp M_\pm(z,x_0,\alpha)^{-1}$ *are necessarily real and at most of first order in the sense that*

$$\mp \lim_{\epsilon\downarrow 0} (i\epsilon\, M_\pm(\lambda+i\epsilon,x_0,\alpha)) \geq 0, \quad \lambda\in\mathbb{R}, \tag{2.61}$$

$$\pm \lim_{\epsilon\downarrow 0} \left(\frac{i\epsilon}{M_\pm(\lambda+i\epsilon,x_0,\alpha)}\right) \geq 0, \quad \lambda\in\mathbb{R}. \tag{2.62}$$

(ii) $\pm M_\pm(z,x_0,\alpha)$ *admit the representations*

$$\pm M_\pm(z,x_0,\alpha) = F_\pm(x_0,\alpha) + \int_{\mathbb{R}} d\Omega_\pm(\lambda,x_0,\alpha)\, \big((\lambda-z)^{-1} - \lambda(1+\lambda^2)^{-1}\big) \tag{2.63}$$

$$= \exp\bigg(C_\pm(x_0,\alpha) + \int_{\mathbb{R}} d\lambda\, \Xi_\pm(\lambda,x_0,\alpha)\big((\lambda-z)^{-1} - \lambda(1+\lambda^2)^{-1}\big)\bigg), \tag{2.64}$$

where

$$F_\pm(x_0,\alpha) = F_\pm(x_0,\alpha)^*, \quad \int_{\mathbb{R}} \|d\Omega_\pm(\lambda,x_0,\alpha)\|_{\mathbb{C}^{m\times m}}\, (1+\lambda^2)^{-1} < \infty, \tag{2.65}$$

$$C_\pm(x_0,\alpha) = C_\pm(x_0,\alpha)^*, \quad 0 \leq \Xi_\pm(\,\cdot\,,x_0,\alpha) \leq I_m \ \text{ a.e.} \tag{2.66}$$

Moreover,

$$\Omega_\pm((\lambda,\mu],x_0,\alpha) = \lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0} \frac{1}{\pi}\int_{\lambda+\delta}^{\mu+\delta} d\nu\, \mathrm{Im}(\pm M_\pm(\nu+i\varepsilon,x_0,\alpha)), \tag{2.67}$$

$$\Xi_\pm(\lambda,x_0,\alpha) = \lim_{\varepsilon\downarrow 0} \pi^{-1}\mathrm{Im}(\ln(\pm M_\pm(\lambda+i\varepsilon,x_0,\alpha))) \text{ for a.e. } \lambda\in\mathbb{R}. \tag{2.68}$$

(iii) Define the $2m\times m$ *matrices*

$$\begin{aligned} U_\pm(z,x,x_0,\alpha) &= \begin{pmatrix} u_{\pm,1}(z,x,x_0,\alpha) \\ u_{\pm,2}(z,x,x_0,\alpha)\end{pmatrix} = \Psi(z,x,x_0,\alpha)\begin{pmatrix} I_m \\ M_\pm(z,x_0,\alpha)\end{pmatrix} \\ &= \begin{pmatrix} \theta_1(z,x,x_0,\alpha) & \phi_1(z,x,x_0,\alpha) \\ \theta_2(z,x,x_0,\alpha) & \phi_2(z,x,x_0,\alpha)\end{pmatrix}\begin{pmatrix} I_m \\ M_\pm(z,x_0,\alpha)\end{pmatrix}, \end{aligned} \tag{2.69}$$

with $\theta_j(z,x,x_0,\alpha)$*, and* $\phi_j(z,x,x_0,\alpha)$*,* $j=1,2$*, defined by* (2.14c)*. Then,*

$$\mathrm{Im}(M_\pm(z,x_0,\alpha)) = \mathrm{Im}(z)\int_{x_0}^{\pm\infty} ds\, U_\pm(z,s,x_0,\alpha)^* A(s) U_\pm(z,s,x_0,\alpha). \tag{2.70}$$

In the Dirac-type context, where $A = I_{2m}$, the m columns of $U_\pm(z,\cdot,x_0,\alpha)$ span $N(z,\pm\infty)$.

Up to this point, we focused exclusively on Hamiltonian systems and neglected the notion of a linear operator associated with (2.2). We did this on purpose as the formalism presented thus far is widely applicable and goes beyond the prime candidates such as Schrödinger and Dirac-type systems. However, in the remainder of this section and for the bulk of the material from Section 3 on, we will focus on the Dirac-type case. Thus, in addition to Hypotheses 2.1–2.3, which are assumed throughout this paper, we introduce the following hypothesis taylored to these occasions.

Hypothesis 2.17. *Assume Hypotheses 2.1 and 2.3 as well as the Dirac-type assumption* (2.4).

Assuming the Dirac-type Hypothesis 2.17, we now describe the associated Dirac-type operator D on $\mathbb{R}$ by first introducing the Green's matrix associated with (2.2) and (2.4). Define the $2m \times 2m$ matrix G by

$$G(z,x,x') = U_{\mp}(z,x,x_0,\alpha_0)[M_-(z,x_0,\alpha_0) - M_+(z,x_0,\alpha_0)]^{-1} U_{\pm}(\overline{z},x',x_0,\alpha_0)^*, \\ \alpha_0 = (I_m\ 0), \quad x \lessgtr x',\ z \in \mathbb{C}\backslash\mathbb{R} \quad (2.71)$$

Next, let $\phi \in L^2(\mathbb{R})^{2m}$ and consider

$$J\psi'(z,x) = (zI_{2m} + B(x))\psi(z,x) + \phi(x), \quad z \in \mathbb{C}\backslash\mathbb{R} \tag{2.72}$$

for a.e. $x \in \mathbb{R}$. Then, as inferred from [62], [64], (2.72) has a unique solution $\psi(z,\cdot\,) \in L^2(\mathbb{R})^{2m} \cap \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^{2m}$ given by

$$\psi(z,x) = \int_{\mathbb{R}} dx'\, G(z,x,x')\phi(x'). \tag{2.73}$$

The Dirac-type operator D in $L^2(\mathbb{R})^{2m}$ associated with the Hamiltonian system (2.2) and (2.4) is then defined by

$$((D-z)^{-1}\psi)(x) = \int_{\mathbb{R}} dx'\, G(z,x,x')\psi(x'), \quad \psi \in L^2(\mathbb{R})^{2m},\ z \in \mathbb{C}\backslash\mathbb{R}. \tag{2.74}$$

Explicitly, one obtains

$$D = J\frac{d}{dx} - B, \tag{2.75}$$
$$\mathrm{dom}(D) = \{\phi \in L^2(\mathbb{R})^{2m} \mid \phi \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^{2m};\ (J\phi' - B\phi) \in L^2(\mathbb{R})^{2m}\},$$

taking into account the limit point property of Dirac-type systems as described in Lemma 2.15. Thus, D is a self-adjoint operator in $L^2(\mathbb{R})^{2m}$.

As described in [62]–[66], the $2m \times 2m$ Weyl-Titchmarsh matrix $M(z,x_0,\alpha_0)$ associated with D is then defined by

$$\begin{aligned} M(z,x_0,\alpha_0) &= \big(M_{j,j'}(z,x_0,\alpha_0)\big)_{j,j'=1,2} \\ &= [G(z,x_0,x_0+0) + G(z,x_0,x_0-0)]/2, \quad z \in \mathbb{C}\backslash\mathbb{R}. \end{aligned} \tag{2.76}$$

Actually, one can replace $\alpha_0 = (I_m\ 0)$ by an arbitrary matrix $\alpha = [\alpha_1\ \alpha_2] \in \mathbb{C}^{m\times 2m}$ satisfying (2.9) and hence introduces

$$M(z,x_0,\alpha) = \big(M_{j,j'}(z,x_0,\alpha)\big)_{j,j'=1,2}, \quad z \in \mathbb{C}\backslash\mathbb{R}, \tag{2.77a}$$
$$M_{1,1}(z,x_0,\alpha) = [M_-(z,x_0,\alpha) - M_+(z,x_0,\alpha)]^{-1}, \tag{2.77b}$$
$$M_{1,2}(z,x_0,\alpha) = 2^{-1}[M_-(z,x_0,\alpha) - M_+(z,x_0,\alpha)]^{-1}[M_-(z,x_0,\alpha) + M_+(z,x_0,\alpha)],$$
$$M_{2,1}(z,x_0,\alpha) = 2^{-1}[M_-(z,x_0,\alpha) + M_+(z,x_0,\alpha)][M_-(z,x_0,\alpha) - M_+(z,x_0,\alpha)]^{-1},$$
$$M_{2,2}(z,x_0,\alpha) = M_{\pm}(z,x_0,\alpha)[M_-(z,x_0,\alpha) - M_+(z,x_0,\alpha)]^{-1}M_{\mp}(z,x_0,\alpha).$$

The basic results on $M(z,x_0,\alpha)$ then read as follows.

Theorem 2.18 ([51], [62], [63], [66], [78]). *Assume Hypothesis 2.17 and suppose that $z \in \mathbb{C}\backslash\mathbb{R}$, $x_0 \in \mathbb{R}$, and that $\alpha \in \mathbb{C}^{m\times 2m}$ satisfies* (2.9)*. Then,*
(i) $M(z,x_0,\alpha)$ is a matrix-valued Herglotz function of rank $2m$ with representation

$$M(z,x_0,\alpha) = F(x_0,\alpha) + \int_{\mathbb{R}} d\Omega(\lambda,x_0,\alpha)\,\big((\lambda-z)^{-1} - \lambda(1+\lambda^2)^{-1}\big), \tag{2.78}$$
$$= \exp\bigg(C(x_0,\alpha) + \int_{\mathbb{R}} d\lambda\,\Upsilon(\lambda,x_0,\alpha)\big((\lambda-z)^{-1} - \lambda(1+\lambda^2)^{-1}\big)\bigg), \tag{2.79}$$

where

$$F(x_0,\alpha)=F(x_0,\alpha)^*, \quad \int_{\mathbb{R}} \|d\Omega(\lambda,x_0,\alpha)\|_{\mathbb{C}^{2m\times 2m}}\,(1+\lambda^2)^{-1}<\infty, \tag{2.80}$$

$$C(x_0,\alpha)=C(x_0,\alpha)^*, \quad 0\le \Upsilon(\,\cdot\,,x_0,\alpha)\le I_{2m} \ \textit{ a.e.} \tag{2.81}$$

Moreover,

$$\Omega((\lambda,\mu],x_0,\alpha)=\lim_{\delta\downarrow 0}\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{\lambda+\delta}^{\mu+\delta} d\nu\,\mathrm{Im}(M(\nu+i\varepsilon,x_0,\alpha)), \tag{2.82}$$

$$\Upsilon(\lambda,x_0,\alpha)=\lim_{\varepsilon\downarrow 0}\pi^{-1}\mathrm{Im}(\ln(M(\lambda+i\varepsilon,x_0,\alpha))) \ \textit{for a.e. } \lambda\in\mathbb{R}. \tag{2.83}$$

(ii) $z\in\mathbb{C}\backslash\mathrm{spec}(D)$ *if and only if* $M(z,x_0,\alpha)$ *is holomorphic near* z.

Here $\mathrm{spec}(T)$ abbreviates the spectrum of a linear operator T.

Next, we explicitly discuss the elementary Dirac-type example where $A=I_{2m}$ and $B=0$.

Example 2.19. Suppose $A=I_{2m}$, $B=0$ and let $\alpha\in\mathbb{C}^{m\times 2m}$ satisfy (2.9). Then,

$$\Theta(z,x,x_0,\alpha)=\begin{pmatrix}\theta_1(z,x,x_0,\alpha)\\ \theta_2(z,x,x_0,\alpha)\end{pmatrix}=\begin{pmatrix}\alpha_1^*\cos(z(x-x_0))+\alpha_2^*\sin(z(x-x_0))\\ \alpha_2^*\cos(z(x-x_0))-\alpha_1^*\sin(z(x-x_0))\end{pmatrix}, \quad z\in\mathbb{C}, \tag{2.84}$$

$$\Phi(z,x,x_0,\alpha)=\begin{pmatrix}\phi_1(z,x,x_0,\alpha)\\ \phi_2(z,x,x_0,\alpha)\end{pmatrix}=\begin{pmatrix}-\alpha_2^*\cos(z(x-x_0))+\alpha_1^*\sin(z(x-x_0))\\ \alpha_1^*\cos(z(x-x_0))+\alpha_2^*\sin(z(x-x_0))\end{pmatrix}, \quad z\in\mathbb{C}, \tag{2.85}$$

$$U_\pm(z,x,x_0,\alpha)=\begin{pmatrix}u_{\pm,1}(z,x,x_0,\alpha)\\ u_{\pm,2}(z,x,x_0,\alpha)\end{pmatrix}=\begin{pmatrix}\alpha_1^*\mp i\alpha_2^*\\ \pm i(\alpha_1^*\mp i\alpha_2^*)\end{pmatrix}\exp(\pm iz(x-x_0)), \quad z\in\mathbb{C}_+, \tag{2.86}$$

$$M_\pm(z,x,\alpha)=\pm iI_m, \quad z\in\mathbb{C}_+. \tag{2.87}$$

Compared to the case of Schrödinger operators, it is remarkable that $M_\pm(z,x,\alpha)$ in (2.87) is, in fact, independent of α. Put differently, in Dirac-type situations, $M_\pm(z,x,\alpha)$ may contain no information on the boundary condition indexed by $\alpha\in\mathbb{C}^{m\times 2m}$.

In Sections 4 and 5 we will also refer to half-line Dirac operators $D_+(\alpha)$ in $L^2([x_0,\infty))^{2m}$ associated with a self-adjoint boundary condition at x_0 indexed by $\alpha\in\mathbb{C}^{m\times 2m}$ satisfying (2.9), and hence briefly introduce

$$D_+(\alpha)=J\frac{d}{dx}-B, \tag{2.88}$$

$$\mathrm{dom}(D_+(\alpha))=\{\phi\in L^2([x_0,\infty))^{2m}\mid \phi\in\mathrm{AC}([x_0,R])^{2m} \text{ for all } R>0; \\ \alpha\phi(x_0)=0;\ (J\phi'-B\phi)\in L^2([x_0,\infty))^{2m}\},$$

taking into account the limit point property of Dirac-type systems at $+\infty$ as described in Lemma 2.15. Thus, $D_+(\alpha)$ is a self-adjoint operator in $L^2([x_0,\infty))^{2m}$. In complete analogy one introduces $D_-(\alpha)$ in $L^2((-\infty,x_0])^{2m}$.

Next, we recall a few formulas in connection with Lagrange's identity needed in the proof of Theorem 5.3 assuming $\alpha\in\mathbb{C}^{m\times 2m}$ satisfies (2.9). Then, explicitly, (2.28) and (2.29) read

$$\theta_2(\bar z,x,x_0,\alpha)^*\theta_1(z,x,x_0,\alpha)-\theta_1(\bar z,x,x_0,\alpha)^*\theta_2(z,x,x_0,\alpha)=0, \tag{2.89}$$

$$\phi_2(\bar{z},x,x_0,\alpha)^*\phi_1(z,x,x_0,\alpha) - \phi_1(\bar{z},x,x_0,\alpha)^*\phi_2(z,x,x_0,\alpha) = 0, \tag{2.90}$$
$$\phi_2(\bar{z},x,x_0,\alpha)^*\theta_1(z,x,x_0,\alpha) - \phi_1(\bar{z},x,x_0,\alpha)^*\theta_2(z,x,x_0,\alpha) = I_m, \tag{2.91}$$
$$\theta_1(\bar{z},x,x_0,\alpha)^*\phi_2(z,x,x_0,\alpha) - \theta_2(\bar{z},x,x_0,\alpha)^*\phi_1(z,x,x_0,\alpha) = I_m, \tag{2.92}$$

and

$$\phi_1(z,x,x_0,\alpha)\theta_1(\bar{z},x,x_0,\alpha)^* - \theta_1(z,x,x_0,\alpha_0)\phi_1(\bar{z},x,x_0,\alpha)^* = 0, \tag{2.93}$$
$$\phi_2(z,x,x_0,\alpha)\theta_2(\bar{z},x,x_0,\alpha)^* - \theta_2(z,x,x_0,\alpha)\phi_2(\bar{z},x,x_0,\alpha)^* = 0, \tag{2.94}$$
$$\phi_2(z,x,x_0,\alpha)\theta_1(\bar{z},x,x_0,\alpha)^* - \theta_2(z,x,x_0,\alpha)\phi_1(\bar{z},x,x_0,\alpha)^* = I_m, \tag{2.95}$$
$$\theta_1(z,x,x_0,\alpha)\phi_2(\bar{z},x,x_0,\alpha)^* - \phi_1(z,x,x_0,\alpha)\theta_2(\bar{z},x,x_0,\alpha)^* = I_m. \tag{2.96}$$

Finally, we note the connection between Φ defined in (2.14b), for different boundary value data $\alpha,\gamma \in \mathbb{C}^{m\times 2m}$ satisfying (2.9), namely

$$\Phi(z,x,x_0,\gamma) = \Phi(z,x,x_0,\alpha)\alpha\gamma^* + \Theta(z,x,x_0,\alpha)\alpha J\gamma^*. \tag{2.97}$$

This connection formula follows by the uniqueness of solutions of (2.2) and by the identity given in (2.13). It is needed in the proof of Theorem 5.3.

3. The Leading Order Term in the Asymptotic Expansion of $M_\pm(z,x,\alpha)$

Assuming Hypothesis 2.17, the principal result proven in this section will be the following leading-order asymptotic result for half-line Weyl-Titchmarsh matrices $M_\pm(z,x_0,\alpha_0)$ associated with the Dirac-type operator (2.75),

$$M_\pm(z,x_0,\alpha_0) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \pm iI_m + o(1). \tag{3.1}$$

Here $\alpha_0 = (I_m\ 0) \in \mathbb{C}^{m\times 2m}$, and $C_\varepsilon \subset \mathbb{C}_+$ denotes the open sector with vertex at zero, symmetry axis along the positive imaginary axis, and opening angle ε, with $0<\varepsilon<\pi/2$.

This particular topic originates with the order result of Hille [57] and the asymptotic formulas of Everitt [30] and of Everitt and Halvorsen [31]. By appealing to the theory of Riccati equations, Atkinson in [9], [10], and [11] obtains results like those of Hille, Everitt, and Halvorsen, both for the Schrödinger case as well as for the scalar-Dirac ($m=1$) case. Through a deeper understanding of the role played by Riccati theory, Atkinson obtains the first order asymptotic expansion of $M_+(z,x,\alpha_0)$ for the matrix-valued Schrödinger case in an unpublished manuscript [12]. Our strategy of proof for (3.1) is patterned after Atkinson's approach which also appears in our recent work on the full asymptotic expansion for $M_+(z,x,\alpha_0)$ in the matrix-valued Schrödinger case [20].

We begin our discussion by noting two additional characterizations for the Weyl disk, $\mathcal{D}(z,c,x_0,\alpha)$, for the general Hamiltonian system (2.2a).

Lemma 3.1. *Assume Hypotheses 2.1 and 2.2. Let $z\in\mathbb{C}\backslash\mathbb{R}$, $c\neq x_0$, and define $U(z,x,x_0,\alpha)$, in terms of $M\in\mathbb{C}^{m\times m}$ by (2.17). Then $M\in\mathcal{D}(z,c,x_0,\alpha)$ if and only if*

$$\sigma(c,x_0,z)\operatorname{Im}(u_1(z,x,x_0,\alpha)^*u_2(z,x,x_0,\alpha)) > 0, \quad x\in[x_0,c), \tag{3.2}$$

or equivalently, if and only if

$$\sigma(c,x_0,z)\operatorname{Im}(u_2(z,x,x_0,\alpha)u_1(z,x,x_0,\alpha)^{-1}) > 0, \quad x\in[x_0,c). \tag{3.3}$$

Moreover, $M \in \mathcal{D}_\pm(z, x_0, \alpha)$ *if and only if* (3.2) *and* (3.3) *hold for* $c = \pm\infty$.

Proof. Let $U(z,x) = U(z,x,x_0,\alpha)$, and let $u_j(z,x) = u_j(z,x,x_0,\alpha)$, $j = 1,2$ with $x \in [x_0, c)$. By (2.26),

$$\begin{aligned} &2\sigma(c,x_0)|\mathrm{Im}(z)| \int_x^c ds\, U(z,s)^* A(s) U(z,s) \\ &= \sigma(x_0,c,z) U(z,s)^*(iJ)U(z,s)\Big|_x^c. \end{aligned} \tag{3.4}$$

By (2.18), this yields

$$\begin{aligned} &2\sigma(c,x_0,z)\mathrm{Im}(u_1(z,x)^* u_2(z,x)) \\ &= -E_c(M) + 2\sigma(c,x_0)|\mathrm{Im}(z)| \int_x^c ds\, U(z,s)^* A(s) U(z,s). \end{aligned} \tag{3.5}$$

The integral expression in (3.5) is strictly positive by Hypothesis 2.2. This yields the equivalence of $-E_c(M) \geq 0$, and hence of $M \in \mathcal{D}(z,c,x_0,\alpha)$, with the condition given in (3.2). The equivalence of (3.2) and (3.3) follows from the observation that

$$\mathrm{Im}(u_2(z,x)u_1(z,x)^{-1}) = (u_1(z,x)^{-1})^* \mathrm{Im}(u_1(z,x)^* u_2(z,x)) u_1(z,x)^{-1}. \tag{3.6}$$

The analogous characterization of $\mathcal{D}_\pm(z,x_0,\alpha)$ now follows from Definition 2.12. □

In Lemma 3.1, $u_j(z,c)$, $j = 1,2$, are well-defined and $E_c(M) = 0$ precisely when $\sigma(c,x_0,z)\mathrm{Im}(u_1(z,c)^* u_2(z,c)) = 0$. A similar statement might not hold for (3.3) since $u_1(z,c,x_0,\alpha)$ might be singular. In part, the latter point motivates the next characterization of the disk.

Lemma 3.2. *Assume Hypotheses 2.1 and 2.2. Let* $z \in \mathbb{C}\backslash\mathbb{R}$, $c \neq x_0$, *and define* $u_j(z,x) = u_j(z,x,x_0,\alpha)$, $j = 1,2$, *by* (2.17). *Then* $M \in \mathcal{D}(z,c,x_0,\alpha)$ *if and only if*

$$u_1(z,x) - i\sigma(c,x_0,z)u_2(z,x) \tag{3.7}$$

is nonsingular for $x \in [x_0, c]$ *and*

$$\begin{aligned} \vartheta(z,x) = \vartheta(z,x,x_0,\alpha) = [u_1(z,x) + i\sigma(c,x_0,z)u_2(z,x)] \times \\ \times [u_1(z,x) - i\sigma(c,x_0,z)u_2(z,x)]^{-1} \end{aligned} \tag{3.8}$$

satisfies

$$I_m - \vartheta(z,x)^*\vartheta(z,x) > 0, \quad x \in [x_0, c), \tag{3.9}$$

with nonnegativity holding at $x = c$. *Moreover,* $M \in \mathcal{D}_\pm(z,x_0,\alpha)$ *if and only if* (3.8) *is well-defined on* $[x_0, \pm\infty)$ *and* (3.9) *holds for* $c = \pm\infty$.

Proof. Let $M \in \mathcal{D}(z,c,x_0,\alpha)$ and suppose that $u_1(z,\xi)v = i\sigma(c,x_0,z)u_2(z,\xi)v$ for $\xi \in [x_0, c]$ and $v \in \mathbb{C}^m$, $v \neq 0$. Then,

$$v^*\sigma(c,x_0,z)\mathrm{Im}(u_1(z,\xi)^* u_2(z,\xi))v = -v^* u_1(z,\xi)^* u_1(z,\xi)v. \tag{3.10}$$

By (3.2), an immediate contradiction results if $\xi \neq c$. However, if $\xi = c$, then either $v^* E_c(M) v > 0$ or $u_j(z,c)v = 0$, $j = 1,2$. In either case, a contradiction results since $E_c(M) \leq 0$ by Definition 2.7 and $U = (u_1^t, u_2^t)^t$ satisfies the first-order system (2.2a). Hence, $\vartheta(z,x)$ is well-defined on $[x_0, c]$. For $x \in [x_0, c)$ and $M \in \mathcal{D}(z,c,x_0,\alpha)$, (3.2) implies that

$$2i\sigma(c,x_0,z)(u_1(z,x)^* u_2(z,x) - u_2(z,x)^* u_1(z,x)) < 0. \tag{3.11}$$

This is equivalent to

$$\begin{aligned}&[u_1(z,x)^* - i\sigma(c,x_0,z)u_2(z,x)^*][u_1(z,x) + i\sigma(c,x_0,z)u_2(z,x)]\\ &\quad < [u_1(z,x)^* + i\sigma(c,x_0,z)u_2(z,x)^*][u_1(z,x) - i\sigma(c,x_0,z)u_2(z,x)]\end{aligned} \tag{3.12}$$

on $[x_0, c)$. Given the nonsingularity of $u_1(z,x) - i\sigma(c,x_0,z)u_2(z,x)$ on $[x_0, c]$, (3.12) implies (3.9), with nonnegativity holding at $x = c$.
Next, let $M \in \mathbb{C}^{m\times m}$, and suppose that $\vartheta(z,x)$, defined by (3.8), is well-defined on $[x_0, c]$, and satisfies (3.9). Then, on $[x_0, c)$, (3.12) and consequently (3.11) follow, which implies that (3.2) holds, and hence that $M \in \mathcal{D}(z, c, x_0, \alpha)$. The analogous characterization of $\mathcal{D}_\pm(z, x_0, \alpha)$ follows from Definition 2.12. ☐

By Lemma 3.1 one notes, for $z \in \mathbb{C}\backslash\mathbb{R}$, that $M \in \mathcal{D}(z, c, x_0, \alpha)$ if and only if

$$V(z,x,x_0,\alpha) = u_2(z,x,x_0,\alpha)u_1(z,x,x_0,\alpha)^{-1}, \quad x \in [x_0, c), \tag{3.13}$$

is well-defined while satisfying

$$\sigma(c,x_0,z)\mathrm{Im}(V(z,x,x_0,\alpha)) > 0, \quad x \in [x_0, c). \tag{3.14}$$

In terms of $V(z,x,x_0,\alpha)$ and by (3.8), one notes that

$$\begin{aligned}\vartheta(z,x,x_0,\alpha) = {}&[I_m + i\sigma(c,x_0,z)V(z,x,x_0,\alpha)] \times\\ &\times [I_m - i\sigma(c,x_0,z)V(z,x,x_0,\alpha)]^{-1}, \quad x \in [x_0, c),\end{aligned} \tag{3.15}$$

is a Cayley-type transformation of $V(z,x,x_0,\alpha)$. In the scalar context, this transformation corresponds to a conformal mapping of the complex upper half-plane to the unit disk. Moreover, defined as it is, $V(z,x,x_0,\alpha)$ satisfies a Riccati differential equation that is associated with the Hamiltonian system (2.2a) while $\vartheta(z,x,x_0,\alpha)$ satisfies a Riccati equation obtained by the Cayley-type transformation (3.15) applied to the differential equation satisfied by $V(z,x,x_0,\alpha)$.

For the Dirac-type case of (2.2a), one observes by a simple calculation that $V(z,x,x_0,\alpha_0)$ is seen to satisfy a particular initial value problem for a Riccati differential equation.

Lemma 3.3. *Assume Hypotheses 2.1, 2.2, and 2.17. Let $\alpha_0 = (I_m\ 0) \in \mathbb{C}^{m\times 2m}$, let $u_j(z,x) = u_j(z,x,x_0,\alpha_0)$, $j = 1,2$, be defined by* (2.17)*, and suppose that $V(z,x,x_0,\alpha_0)$ is well-defined by* (3.13)*. Then, $V(z,\cdot) = V(z,\cdot,x_0,\alpha_0)$ satisfies,*

$$V'(z,x) + zV(z,x)^2 + V(z,x)B_{2,2}(x)V(z,x) + B_{1,2}(x)V(z,x) + V(z,x)B_{2,1}(x)$$
$$+ B_{1,1}(x) + zI_m = 0, \tag{3.16a}$$
$$V(z,x_0) = M, \tag{3.16b}$$

where $B_{j,k} \in \mathbb{C}^{m\times m}$, $j,k = 1,2$, are defined in (2.1d).

Hence, by Lemma 3.1, the associated relations (3.13) and (3.14), and the uniqueness of solutions for (3.16), we obtain the following result for the Dirac-type case.

Theorem 3.4. *Assume Hypotheses 2.1, 2.2, and 2.17, and let $\alpha_0 = (I_m\ 0) \in \mathbb{C}^{m\times 2m}$. Then, $M \in \mathcal{D}(z,c,x_0,\alpha_0)$ if and only if the initial value problem given by* (3.16) *has a solution, $V(z,\cdot)$, well-defined and satisfying*

$$\sigma(c,x_0,z)\mathrm{Im}(V(z,x)) > 0, \quad x \in [x_0, c). \tag{3.17}$$

Moreover, $M \in \mathcal{D}_\pm(z,x_0,\alpha_0)$ if and only if (3.17) *holds for $c = \pm\infty$.*

Remark 3.5. An important consequence of Theorem 3.4 and the uniqueness of solutions for (3.16) is that solution trajectories for (3.16), which satisfy (3.17), consist of elements of Weyl disks; that is,

$$V(z,x,x_0,\alpha_0)\in\mathcal{D}(z,c,x,\alpha_0),\quad x\in[x_0,c). \tag{3.18}$$

Given the characterization of $\mathcal{D}(z,c,x_0,\alpha_0)$ in Defintion 2.7A, for each $x\in[x_0,c)$ there is a $\beta\in\mathbb{C}^{m\times 2m}$ with $\sigma(c,x_0,z)\operatorname{Im}(\beta_2\beta_1^*)\geq 0$, such that

$$V(z,x,x_0,\alpha_0)=M(z,c,x,\alpha_0,\beta). \tag{3.19}$$

It is in this sense that we let $M(z,c,x,\alpha_0)$ denote our solution of the initial value problem (3.16) that satisfies (3.17). Analogously,

$$V(z,x,x_0,\alpha_0)\in\mathcal{D}_\pm(z,x,\alpha_0),\quad x\in[x_0,\pm\infty), \tag{3.20}$$

for trajectories of (3.16) that satisfy (3.17) for $c=\pm\infty$. Hence, in this sense, we let $M_\pm(z,x,\alpha_0)$ denote those solutions of (3.16) that satisfy (3.17) for $c=\pm\infty$. However, by Lemma 2.15, our Dirac system is in the limit point case at $\pm\infty$. Each $\mathcal{D}_\pm(z,x,\alpha_0)$ consists of a unique matrix, and thus $M_\pm(z,x,\alpha_0)$ describes *unique* trajectories for (3.16a). This contrasts with the matrix-valued Schrödinger case considered in [20] where there are as many trajectories, each denoted by either $M_+(z,x,\alpha_0)$ or $M_-(z,x,\alpha_0)$, as there are matrices in a given initial disk $\mathcal{D}_\pm(z,x_0,\alpha_0)$.

Now for the Dirac-type case (2.4) with $\alpha_0=(I_m\ 0)\in\mathbb{C}^{m\times 2m}$, with $\vartheta(z,x)=\vartheta(z,x,x_0,\alpha_0)$ defined in (3.8) and (3.15), and with $x\in[x_0,c)$, one concludes that

$$\vartheta(z,x)[u_1(z,x)-i\sigma(c,x_0,z)u_2(z,x)]=u_1(z,x)+i\sigma(c,x_0,z)u_2(z,x), \tag{3.21}$$

and hence that

$$I_m+\vartheta(z,x)=2u_1(z,x)[u_1(z,x)-i\sigma(c,x_0,z)u_2(z,x)]^{-1}, \tag{3.22a}$$

$$I_m-\vartheta(z,x)=-2i\sigma(c,x_0,z)u_2(z,x)[u_1(z,x)-i\sigma(c,x_0,z)u_2(z,x)]^{-1}. \tag{3.22b}$$

Differentiating (3.21) one obtains

$$\begin{aligned}\vartheta'(u_1-i\sigma u_2)&=(I_m-\vartheta)(zu_2+B_{2,1}u_1+B_{2,2}u_2)\\&\quad+i\sigma(I_m+\vartheta)(-zu_1-B_{1,1}u_1-B_{1,2}u_2).\end{aligned} \tag{3.23}$$

By (3.22) one concludes that $\vartheta(z,\cdot,x_0,\alpha_0)$ satisfies the initial value problem given by

$$\begin{aligned}\vartheta'(z,x)&=\frac{1}{2}\begin{pmatrix}I_m+\vartheta(z,x)^t\\ I_m-\vartheta(z,x)^t\end{pmatrix}^t\times\\&\quad\times\begin{pmatrix}-i\sigma(c,x_0,z)(zI_m+B_{1,1}(x)) & B_{1,2}(x)\\ B_{2,1}(x) & i\sigma(c,x_0,z)(zI_m+B_{2,2}(x))\end{pmatrix}\times\\&\quad\times\begin{pmatrix}I_m+\vartheta(z,x)\\ I_m-\vartheta(z,x)\end{pmatrix},\end{aligned} \tag{3.24a}$$

$$\vartheta(z,x_0)=(I_m+i\sigma(c,x_0,z)M)(I_m-i\sigma(c,x_0,z)M)^{-1}, \tag{3.24b}$$

where $B_{j,k}\in\mathbb{C}^{m\times m}$, $j,k=1,2$, satisfy Hypothesis 2.1.

By Lemma 3.2 and the uniqueness of solutions for (3.24), one obtains the following result in the Dirac-type case (2.4).

Theorem 3.6. *Assume Hypothesis 2.17. Then $M \in \mathcal{D}(z, c, x_0, \alpha_0)$ if and only if the initial value problem given by* (3.24) *has a solution, $\vartheta(z, \cdot)$ which is well-defined on $[x_0, c]$ and satisfies*

$$I_m - \vartheta(z, x)^* \vartheta(z, x) > 0, \quad x \in [x_0, c). \tag{3.25}$$

Moreover, $M \in \mathcal{D}_\pm(z, x_0, \alpha_0)$ if and only if (3.25) *holds for $c = \pm\infty$.*

Given the positivity present in (3.25), we note the exact correspondence which exists, by (3.15), between solutions of (3.16) that satisfy (3.17) and those solutions of (3.24) that satisfy (3.25). In particular, given Remark 3.5, we rewrite (3.15) as

$$\begin{aligned} \vartheta(z, x, x_0, \alpha_0) = [I_m + i\sigma(c, x_0, z)M(z, c, x, \alpha_0)] \times \\ \times [I_m - i\sigma(c, x_0, z)M(z, c, x, \alpha_0)]^{-1}, \quad x \in [x_0, c), \end{aligned} \tag{3.26}$$

Moreover, our Dirac system is in the limit point case at $\pm\infty$. Consequently, there are unique solutions of (3.24), $\vartheta_\pm(z, \cdot, x_0, \alpha_0)$, $z \in \mathbb{C}\backslash\mathbb{R}$, which satisfy (3.25) for $c = \pm\infty$, and which correspond to the unique solutions of (3.16), $M_\pm(z, x, \alpha_0)$, which satisfy (3.17) for $c = \pm\infty$; specifically,

$$\vartheta_\pm(z, x, x_0, \alpha_0) = [I_m \pm i\sigma(z)M_\pm(z, x, \alpha_0)][I_m \mp i\sigma(z)M_\pm(z, x, \alpha_0)]^{-1}. \tag{3.27}$$

These relationships form the basis for the analysis to follow. The asymptotic result (3.1) is obtained by an analysis of the corresponding asymptotic behavior for all solutions $\vartheta(z, \cdot, x_0, \alpha_0)$ described in (3.24), these include among them the particular solutions $\vartheta_\pm(z, \cdot, x_0, \alpha_0)$. Thus asymptotic behavior is deduced for all corresponding solutions $M(z, c, \cdot, \alpha_0)$ of (3.16) which include among them the solutions $M_\pm(z, \cdot, \alpha_0)$. The advantage of this approach comes from the compactification inherent in the Cayley-type transformation (3.26), and the resulting boundedness of the solutions as a consequence of (3.25).

We pause for a moment to address, in the following remark, a point raised by us in [20] for the matrix-valued Schrödinger case described in (2.5).

Remark 3.7. With $u_j(z, x) = u_j(z, x, x_0, \alpha)$, $j = 1, 2$, defined in (2.17) for the general Hamiltonian system (2.2a), an analog to Lemma 3.2 for the characterization of $\mathcal{D}(z, c, x_0, \alpha)$ is obtained by replacing the expression in (3.7) with

$$u_1(z, x) - i|z|^{-1/2}\sigma(c, x_0, z)u_2(z, x), \tag{3.28}$$

and by replacing the definition for $\vartheta(z, x) = \vartheta(z, x, x_0, \alpha)$ given in (3.8) with

$$\begin{aligned} \vartheta(z, x) =& (u_1(z, x) + i|z|^{-1/2}\sigma(c, x_0, z)u_2(z, x)) \times \\ & \times (u_1(z, x) - i|z|^{-1/2}\sigma(c, x_0, z)u_2(z, x))^{-1}. \end{aligned} \tag{3.29}$$

Specific to the matrix-valued Schrödinger case, we obtain analogs of Lemma 3.3, Theorem 3.4, and Theorem 3.6 by replacing equation (3.16a) with

$$V'(z, x) + V(z, x)^2 - Q(x) + zI_m = 0 \tag{3.30}$$

and by replacing the equations in (3.24) with

$$\begin{aligned} \vartheta'(z, x) = \sigma(c, x_0, z)\frac{1}{2}\begin{pmatrix} I_m + \vartheta(z, x)^t \\ I_m - \vartheta(z, x)^t \end{pmatrix}^t \begin{pmatrix} -i|z|^{-1/2}(zI_m - Q(x)) & 0 \\ 0 & i|z|^{-1/2}I_m \end{pmatrix} \times \\ \times \begin{pmatrix} I_m + \vartheta(z, x) \\ I_m - \vartheta(z, x) \end{pmatrix}, \end{aligned} \tag{3.31a}$$

$$\vartheta(z, x_0) = (I_m + i|z|^{-1/2}\sigma(c, x_0, z)M)(I_m - i|z|^{-1/2}\sigma(c, x_0, z)M)^{-1}. \tag{3.31b}$$

$\mathcal{D}^{\mathcal{R}}(z,c,x_0,\alpha_0)$ was defined in [20] to be the set of those $M \in \mathbb{C}^{m\times m}$ for which the intial value problem given by (3.31) has a solution, $\vartheta(z,x)$, which is well-defined on $[x_0,c]$ and satisfies (3.25). In [20] we showed that $\mathcal{D}(z,c,x_0,\alpha_0) \subseteq \mathcal{D}^{\mathcal{R}}(z,c,x_0,\alpha_0)$. This was sufficient for the subsequent analysis in [20]. However, as the analog of Theorem 3.6 now shows, one actually has equality of the two disks in [20], that is,

$$\mathcal{D}(z,c,x_0,\alpha_0) = \mathcal{D}^{\mathcal{R}}(z,c,x_0,\alpha_0). \tag{3.32}$$

To obtain a proof of (3.1) for the Dirac-type case, we adapt an approach due to Atkinson [12] for proving a result analogous to (3.1) for the matrix-valued Schrödinger case (cf., e.g., [20, Theorem 3.1]) In light of Remark 3.12, we begin by restricting our attention to $z \in \mathbb{C}_+$, and as in the previous discussion, take $\alpha_0 = (I_m\ 0) \in \mathbb{C}^{m\times 2m}$.

First we introduce two systems related to (3.24) by means of a change of variables. Let

$$\varphi(z,t) = \vartheta(z,x), \qquad t = (x-x_0)|z|, \qquad x \in [x_0,c). \tag{3.33}$$

With this change, (3.24) becomes

$$\begin{aligned}\varphi'(z,t) = \frac{1}{2}|z|^{-1}\begin{pmatrix} I_m+\varphi(z,t)^t \\ I_m-\varphi(z,t)^t\end{pmatrix}^t \begin{pmatrix} \mp i(zI_m+\widetilde B_{1,1}(t)) & \widetilde B_{1,2}(t) \\ \widetilde B_{2,1}(t) & \pm i(zI_m+\widetilde B_{2,2}(t))\end{pmatrix} \times \\ \times \begin{pmatrix} I_m+\varphi(z,t) \\ I_m-\varphi(z,t)\end{pmatrix}.\end{aligned} \tag{3.34a}$$

With $M = M(z,c,x_0,\alpha_0) \in \mathcal{D}(z,c,x_0,\alpha_0)$ (3.24b) becomes

$$\varphi(z,0) = (iI_m \mp M(z,c,x_0,\alpha_0))(iI_m \pm M(z,c,x_0,\alpha_0))^{-1}, \tag{3.34b}$$

and (3.25) becomes

$$\varphi(z,t)^*\varphi(z,t) < I_m \qquad t \in [0,(c-x_0)|z|), \tag{3.34c}$$

where in (3.34a),

$$\widetilde B_{j,k}(t) = B_{j,k}(x_0+t|z|^{-1}), \quad j,k=1,2. \tag{3.34d}$$

In the complete system (3.34), one now has a set of conditions equivalent to system (3.24) and (3.25).

We recall that $C_\varepsilon \subset \mathbb{C}_+$ represents the open sector with vertex at zero, symmetry axis along the positive imaginary axis, and opening angle ε, with $0<\varepsilon<\pi/2$. Next, consider a sequence, $z_n \in \mathbb{C}_\varepsilon$, $n \in \mathbb{N}$, such that $|z_n| \to \infty$ as $n \to \infty$ and such that

$$0 < \varepsilon < \delta_n = \arg(z_n) < \pi - \varepsilon. \tag{3.35}$$

By choosing an appropriate subsequence, we may assume that

$$\delta_n \to \delta \in [\varepsilon, \pi-\varepsilon]. \tag{3.36}$$

Let $\varphi(z_n,t)$ denote a corresponding sequence of functions that satisfy (3.34a) and (3.34c), with initial data, $\varphi(z_n,0)$, defined by (3.34b) for a sequence of points $M(z_n,c,x_0,\alpha_0)$, where each $M(z_n,c,x_0,\alpha_0)$ is chosen to be an element of the disk $\mathcal{D}(z_n,c,x_0,\alpha_0)$. Note that as $z_n \to \infty$, the intervals described in (3.34c) eventually cover all compact subintervals of $\mathbb{R}_+$. Given the uniform boundedness of $\varphi_n(t) = \varphi(z_n,t)$ described in (3.34c), we assume, upon passing to an appropriate subsequence still denoted by $\varphi_n(0)$, that

$$\varphi_n(0) = \varphi(z_n,0) \to \varphi_\pm(\delta), \text{ for } \pm(c-x_0) > 0 \text{ as } n \to \infty, \tag{3.37}$$

and as a consequence, that

$$\varphi_\pm(\delta)^*\varphi_\pm(\delta) \le I_m. \tag{3.38}$$

With $\varphi_\pm(\delta)$ defined in (3.37) as $|z_n| \to \infty$, we consider limiting systems associated with (3.34):

$$\eta_\pm'(t) = \frac{1}{2}\begin{pmatrix} I_m + \eta_\pm(t)^t \\ I_m - \eta_\pm(t)^t \end{pmatrix}^t \begin{pmatrix} \mp ie^{i\delta} I_m & 0 \\ 0 & \pm ie^{i\delta} I_m \end{pmatrix} \begin{pmatrix} I_m + \eta_\pm(t) \\ I_m - \eta_\pm(t) \end{pmatrix}, \quad \pm t \ge 0, \tag{3.39a}$$

$$\eta_\pm(0) = \varphi_\pm(\delta). \tag{3.39b}$$

Theorem 3.8. *Assume Hypothesis 2.17. Then the solution $\eta_\pm$ of* (3.39) *satisfies*

$$\eta_\pm(t)^*\eta_\pm(t) \le I_m, \quad t \in [0, \pm\infty). \tag{3.40}$$

Moreover, the solutions $\varphi_n = \varphi(z_n, \cdot)$ of (3.34) *converge to $\eta_\pm$ uniformly on $[0, \pm T]$ for every $T > 0$, as $n \to \infty$.*

Proof. In this proof, we consider only the case corresponding to $t \ge 0$, that is, $\eta_+(0) = \varphi_+(\delta)$ in (3.39b). The other case follows in a similar manner. For this reason, we let $\eta(t) = \eta_+(t)$ in the remaining discussion. We also let $T \in \mathbb{R}_+$ be the greatest value such that (3.40) holds for $t \in [0, T]$ and show that (3.40) must hold for some $[0, T']$ with $T' > T$, thus proving $T = \infty$.
The solution of (3.39), η, presumed to be defined on $[0, T]$, can be continued onto some $[0, T']$ with $T' > T$; η then satisfies

$$\eta(t)^*\eta(t) \le \kappa^2 I_m \tag{3.41}$$

for $0 \le t \le T'$ and for some $\kappa \ge 1$.
For brevity, let $\varphi_n'(t) = G_n(\varphi_n, t)$ denote (3.34a) with $z = z_n$, and let $\eta'(t) = H(\eta, t)$ denote (3.39a) in the following. Integrating (3.39a) and (3.34a), one obtains

$$\begin{aligned} \varphi_n(t) - \eta(t) = \varphi_n(0) - \varphi_0(\delta) + \int_0^t ds\{G_n(\eta, s) - H(\eta, s)\} \\ + \int_0^t ds\{G_n(\varphi_n, s) - G_n(\eta, s)\}. \end{aligned} \tag{3.42}$$

We note that

$$\begin{aligned} G_n(\eta, s) - H(\eta, s) = \frac{1}{2}i(e^{i\delta} - e^{i\delta_n})(I_m + \eta(s))^2 - \frac{1}{2}i(e^{i\delta} - e^{i\delta_n})(I_m - \eta(s))^2 + \\ + \sum_{j,k=1}^{2} F_{j,k}(z_n, s), \end{aligned} \tag{3.43}$$

where,

$$F_{1,1}(z_n, s) = -\frac{1}{2}i|z_n|^{-1}(I_m + \eta(s))\widetilde{B}_{1,1}(s)(I_m + \eta(s)), \tag{3.44a}$$

$$F_{2,2}(z_n, s) = \frac{1}{2}i|z_n|^{-1}(I_m - \eta(s))\widetilde{B}_{2,2}(s)(I_m - \eta(s)), \tag{3.44b}$$

$$F_{1,2}(z_n, s) = \frac{1}{2}i|z_n|^{-1}(I_m + \eta(s))\widetilde{B}_{1,2}(s)(I_m - \eta(s)), \tag{3.44c}$$

$$F_{2,1}(z_n, s) = \frac{1}{2}i|z_n|^{-1}(I_m - \eta(s))\widetilde{B}_{2,1}(s)(I_m + \eta(s)). \tag{3.44d}$$

Thus, for $t \in [0, T']$, (3.41) implies that as $n \to \infty$

$$|e^{i\delta} - e^{i\delta_n}| \int_0^t \|I_m \pm \eta(s)\|^2_{\mathbb{C}^{m\times m}} ds = o(1), \tag{3.45}$$

and together with (3.33) and (3.34d) that

$$\int_0^t \|F_{j,k}(s)\|_{\mathbb{C}^{m\times m}} ds = O\bigg(\int_{x_0}^{x_0+t|z_n|^{-1}} \|\widetilde{B}_{j,k}(s)\|_{\mathbb{C}^{m\times m}} ds \bigg) = o(1). \tag{3.46}$$

(Here $\|\cdot\|_{\mathbb{C}^{m\times m}}$ denotes a norm on $\mathbb{C}^{m\times m}$.) Hence, by (3.43)–(3.46), one infers that for $t \in [0, T']$ and as $n \to \infty$,

$$\int_0^t \{G_n(\eta, s) - H(\eta, s)\} ds = o(1). \tag{3.47}$$

Next, one notes that

$$G_n(\varphi_n, s) - G_n(\eta, s) = 2ie^{i\delta_n}(\eta(s) - \varphi_n(s)) + \sum_{j,k=1}^{2} K_{j,k}(z_n, s), \tag{3.48}$$

where

$$K_{1,1}(z_n, s) = \frac{-i}{2}|z_n|^{-1}\{(I_m + \varphi_n)B_{1,1}(s)(\varphi_n - \eta) + (\varphi_n - \eta)B_{1,1}(s)(I_m + \eta)\}, \tag{3.49a}$$

$$K_{2,2}(z_n, s) = \frac{i}{2}|z_n|^{-1}\{(I_m - \varphi_n)B_{2,2}(s)(\eta - \varphi_n) + (\eta - \varphi_n)B_{2,2}(s)(I_m - \eta)\}, \tag{3.49b}$$

$$K_{1,2}(z_n, s) = \frac{1}{2}|z_n|^{-1}\{(I_m + \varphi_n)B_{1,2}(s)(\eta - \varphi_n) + (\varphi_n - \eta)B_{1,2}(s)(I_m - \eta)\}, \tag{3.49c}$$

$$K_{2,1}(z_n, s) = \frac{1}{2}|z_n|^{-1}\{(I_m - \varphi_n)B_{2,1}(s)(\varphi_n - \eta) + (\eta - \varphi_n)B_{2,1}(s)(I_m + \eta)\}. \tag{3.49d}$$

By (3.38) and (3.41), for $s \in [0, T']$,

$$\|I_m \pm \varphi_n(s)\|_{\mathbb{C}^{m\times m}} \le 2, \qquad \|I_m \pm \eta(s)\|_{\mathbb{C}^{m\times m}} \le \kappa + 1, \tag{3.50}$$

and hence by (3.48)–(3.50),

$$\begin{aligned} &\|G_n(\varphi_n, s) - G_n(\eta, s)\|_{\mathbb{C}^{m\times m}} \\ &\quad \le \|\eta(s) - \varphi_n(s)\|_{\mathbb{C}^{m\times m}} \bigg\{ 2 + \frac{|z_n|^{-1}}{2}(3 + \kappa) \sum_{j,k=1}^{2} \|\widetilde{B}_{j,k}(s)\|_{\mathbb{C}^{m\times m}} \bigg\}. \end{aligned} \tag{3.51}$$

Of course, by (3.37) as $n \to \infty$,

$$\phi_n(0) - \phi_+(\delta) = o(1). \tag{3.52}$$

Thus, by (3.46), (3.51) and (3.52), one concludes for $t \in [0, T']$ and as $n \to \infty$, that

$$\begin{aligned} &\|\varphi_n(t) - \eta(t)\|_{\mathbb{C}^{m\times m}} \le o(1) \\ &\quad + \int_0^t \|\varphi_n(s) - \eta(s)\|_{\mathbb{C}^{m\times m}} \bigg\{ 2 + \frac{|z_n|^{-1}}{2}(3 + \kappa) \sum_{j,k=1}^{2} \|\widetilde{B}_{j,k}(s)\|_{\mathbb{C}^{m\times m}} \bigg\} ds. \end{aligned} \tag{3.53}$$

Gronwall's inequality applied to (3.53) together with a consideration of the effect of the variable change (3.33), as illustrated in (3.46), yields

$$\varphi_n(t) - \eta(t) \to 0 \;\text{ as }\; n \to \infty \tag{3.54}$$

uniformly for $t \in [0, T']$. Thus by (3.34c), the contradiction results that for all $t \in [0, T']$, η satisfies (3.40). □

What solutions of (3.39) satisfy (3.40)?

Lemma 3.9. *Assume Hypothesis 2.17. If $\eta_\pm$ is a solution of* (3.39a) *which satisfies* (3.40)*, then*

$$0 = \eta_\pm(t), \quad t \in [0, \pm\infty). \tag{3.55}$$

Proof. We note that (3.39a) is equivalent to (3.34a) with $\widetilde{B} = 0$. By the variable change (3.33), (3.39a) is also equivalent to (3.24a) with $B = 0$. Next, we recall the connection between the Riccati-type equations (3.24a), and (3.16a) by means of the Cayley transformation (3.26). Solution matrices of (3.39a) which statisfy (3.40) at $t = 0$ thus correspond to solution matrices, $V(z, \cdot)$, of (3.16a) for which $\mathrm{Im}(V(z, x_0)) \geq 0$. Moreover, solutions of (3.16a) for which for which $\mathrm{Im}(V(z, x_0)) \geq 0$ are obtainable from solutions of (2.2a), with $B = 0$, by means of (2.17) with $\mathrm{Im}(M) \geq 0$. Thus, by utilizing this connection between explicit exponential solutions of (2.2a) with $B = 0$ and solutions of the Riccati-type equation (3.16a), and by performing on the resulting solution of (3.16a) the conformal mapping (3.26) followed by the variable transformation (3.33), one obtains the following solution for (3.34a),

$$\varphi(z, t) = (iI_m \mp M)(iI_m \pm M)^{-1} \exp(\mp 2ite^{i\delta}), \tag{3.56}$$

for $\pm t \geq 0$, $\mathrm{Im}(\pm M) \geq 0$, and $z \in \mathbb{C}_+$. By hypothesis, $0 < \delta < \pi$. Thus the exponential term in (3.56) will result in

$$||\varphi(z, t)||_{\mathbb{C}^{m\times m}} > 1 \;\text{ as } t \to \pm\infty \tag{3.57}$$

unless

$$M = \pm iI_m, \tag{3.58}$$

thus implying (3.55). □

One then obtains the following result.

Corollary 3.10. *With $\phi_\pm(\delta)$ defined in* (3.37)*, $\eta_\pm(0) = \phi_\pm(\delta) = 0$.*

For $M(z_n, c, x_0, \alpha_0) \in \mathcal{D}(z_n, c, x_0, \alpha_0)$, it follows by (3.34b), (3.37), and Corollary 3.10 that

$$[iI_m \mp M(z_n, c, x_0, \alpha_0)][iI_m \pm M(z_n, c, x_0, \alpha_0)]^{-1} = o(1), \qquad \pm(c - x_0) > 0, \tag{3.59}$$

as $n \to \infty$. Hence one infers, for elements of $\mathcal{D}(z_n, c, x_0, \alpha_0)$, that

$$M(z_n, c, x_0, \alpha_0) = \pm iI_m + o(1), \qquad \pm(c - x_0) > 0, \tag{3.60}$$

as $|z| \to \infty$ in C_ε. This proves (3.1). Actually, (3.60) is a statement for all elements of $\mathcal{D}(z, c, x_0, \alpha_0)$ including the particular element $M_\pm(z, x_0, \alpha_0)$, for $\pm(c - x_0) > 0$.

In (3.1) an asymptotic expansion is given that is uniform with respect to $\arg(z)$ for $|z| \to \infty$ in C_ε. We now vary the reference point, x_0, and observe that the

asymptotic expansion in (3.1) is also uniform with respect to x_0 whenever x_0 is confined to a compact subset of $\mathbb{R}$.

Theorem 3.11. *Assume Hypothesis 2.17. Let* $\alpha_0 = (I_m\ 0) \in \mathbb{C}^{m\times 2m}$*, and denote by* $C_\varepsilon \subset \mathbb{C}_+$ *the open sector with vertex at zero, symmetry axis along the positve imaginary axis and opening angle* ε*, with* $0 < \varepsilon < \pi/2$*. Let* $M_\pm(z, x_0, \alpha_0)$ *be the unique elements of the limit disks* $\mathcal{D}_\pm(z, x_0, \alpha_0)$ *for the Dirac system given by* (2.2) *and* (2.4)*. Then,*

$$M_\pm(z, x, \alpha_0) \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} \pm i I_m + o(1) \tag{3.61}$$

uniformly with respect to $\arg(z)$*, for* $|z| \to \infty$ *in* C_ε*, and uniformly with respect to* x*, as long as* x *varies in compact subsets of* $[x_0, \pm\infty)$*.*

Proof. We note that the system (3.39) is independent of the reference point x_0. Next, we recall that δ, defined in (3.36) is determined by an apriori choice of the sequence z_n, subject only to z_n being in C_ε (c.f. (3.35)). Moreover, we note that $\varphi_\pm(\delta)$, defined as a limit in (3.37), described explicity in Corollary 3.10, and which gives solutions of (3.39) satisfying (3.40) for $t \in [0, \pm\infty)$, is also independent of the reference point x_0. Thus, had we chosen a different point of reference, $x_0' \neq x_0$, at the start, the asymptotic analysis begun in Theorem 3.8 and continued through (3.59), would remain the same after the variable change in (3.33), except for the integral expression in (3.46) in which x_0 would be replaced by x_0'. However, given the local integrability assumption on B in Hypothesis 2.1, one concludes that this integral expression is uniformly continuous with respect to x_0 whenever x_0 is confined to a compact subset of $\mathbb{R}$. Thus (3.46), and consequently (3.54), are uniform with respect to t and with respect to x_0 whenever both are confined to compact subsets of $\mathbb{R}$. Consequently, (3.59) holds for elements $\mathcal{D}(z, c, x_0, \alpha_0)$, that this asymptotic expansion is uniform with respect to $\arg(z)$ for $|z| \to \infty$ in C_ε, and that it is uniform with respect to x_0 when x_0 is confined to compact subsets of $\mathbb{R}$. □

Remark 3.12. (i) In the special case $m = 1$, the leading-order asymptotics (3.61) was published by Everitt, Hinton, and Shaw [32] in 1983. For asymptotic estimates of Weyl solutions in the case $m = 1$ we refer to [97].
(ii) A comparison of (3.61) with (2.60) then proves that the leading-order asymptotic behavior (3.61) is in fact independent of the boundary condition at x_0 indexed by α, that is,

$$M_\pm(z, x_0, \alpha) \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} \pm i I_m + o(1) \tag{3.62}$$

for any α satisfying the conditions stated in (2.9). In the scalar case $m = 1$ this fact had been noticed in [32]. This boundary condition independence of the leading-order asymptotic behavior of $M_\pm(z, x_0, \alpha)$ is in sharp contrast to the case of matrix-valued Schrödinger operators (see, e.g., [20]). Moreover, regarding the conclusion of Theorem 3.11, no generality is lost by assuming that $C_\varepsilon \subset \mathbb{C}_+$ because of (2.57).

4. Higher Order Terms in the Asymptotic Expansion of $M_\pm(z, x, \alpha)$

In this section we shall prove one of our principal results of this paper, the asymptotic high-energy expansion of $M_+(z, x, \alpha_0)$ to arbitrarily high orders in sectors of the type $C_\varepsilon \subset \mathbb{C}_+$ as defined in Theorem 3.11.

Throughout this section we choose $z \in \mathbb{C}_+$. We also recall the following notion: $x \in [a, b)$ (resp., $x \in (a, b]$) is called a right (resp., left) Lebesgue point of an element $q \in L^1((a, b))$, $a < b$ if $\int_0^\varepsilon dx' \, |q(x+x') - q(x)| = o(\varepsilon)$ (resp., $\int_0^\varepsilon dx' \, |q(x-x') - q(x)| = o(\varepsilon)$) as $\varepsilon \downarrow 0$. Similarly, $x \in (a, b)$ is called a Lebesgue point of $q \in L^1((a, b))$ if $\int_{-\varepsilon}^\varepsilon dx' \, |q(x + x') - q(x)| = o(\varepsilon)$ as $\varepsilon \downarrow 0$. The set of all such points is then denoted the right (resp., left) Lebesgue set of q on $[a, b]$ in the former case and simply the Lebesgue set of q on $[a, b]$ in the latter case. The analogous notions are applied to $2m \times 2m$ matrices $B \in L^1((a, b))^{2m\times 2m}$ by simultaneously considering all $4m^2$ entries of B. The right (resp., left) Lebesgue set of B on $[a, b]$ is then simply the intersection of the right (resp., left) Lebesgue sets of $B_{j,k}$ for all $1 \leq j, k \leq 2m$, and similarly for the Lebesgue set of B, etc.

Finally, we need one more ingredient, recently proven by Rybkin [101, Lemma 3] using appropriate maximal functions. Let $q \in L^1((x_0, \infty))$, $\operatorname{supp}(q) \subseteq [x_0, x_0 + R]$ for some $R > 0$, and suppose $x \in [x_0, x_0 + R]$ is a right Lebesgue point of q. Then

$$\int_x^{x_0+R} dx' \, q(x') \exp(2iz(x' - x)) \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} -\frac{q(x)}{2iz} + o\big(|z|^{-1}\big). \tag{4.1}$$

An alternative proof of (4.1) follows from [117, Theorem I.13], which implies

$$\lim_{\substack{|z|\to\infty \\ z\in C_\varepsilon}} z^{-1} \int_x^{x_0+R} dx' \, |q(x') - q(x)| \exp(2iz(x' - x)) = 0 \tag{4.2}$$

for any right Lebesgue point x of q.

We start with the simpler case where B has compact support contained in some interval $[x_0, y_0]$. Below in (4.3) and in analogous formulas in this section, $\|\cdot\|_{\mathbb{C}^{\ell\times\ell}}$ denotes a norm in $\mathbb{C}^{\ell\times\ell}$.

Lemma 4.1. *Fix $x_0, y_0 \in \mathbb{R}$ with $y_0 > x_0$ and let $x \geq x_0$. Suppose $A = I_{2m}$, $B \in L^1([x_0, x_0 + R])^{2m\times 2m}$ for all $R > 0$, $B = B^*$ a.e. on (x_0, ∞). In addition, assume that B has compact support contained in $[x_0, y_0]$, that $B^{(N-1)} \in L^1([x_0, y_0])^{2m\times 2m}$ for some $N \in \mathbb{N}$, that x is a right Lebesgue point of $B^{(N-1)}$, and that*

$$\begin{aligned} &\operatorname*{ess\,sup}_{y\in[x_0,y_0]} \left\| \int_y^{y_0} dx' \, B^{(N-1)}(x') \exp(2iz(x' - y)) + \frac{1}{2iz} B^{(N-1)}(y) \right\|_{\mathbb{C}^{2m\times 2m}} \\ &\underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} o\big(|z|^{-1}\big). \end{aligned} \tag{4.3}$$

If $N = 1$, suppose in addition $B_{k,k'} B_{\ell,\ell'} \in L^1([x_0, y_0])^{m\times m}$ for all $k, k', \ell, \ell' \in \{1, 2\}$. Let $\alpha_0 = (I_m \ 0) \in \mathbb{C}^{m\times 2m}$ and denote by $M_+(z, x, \alpha_0)$, $x \geq x_0$, the unique Weyl-Titchmarsh matrix associated with the half-line Dirac-type operator $D_+(\alpha_0)$ in (2.88). Then, as $|z| \to \infty$ in C_ε, $M_+(z, x, \alpha_0)$ has an asymptotic expansion of the form

$$M_+(z, x, \alpha_0) \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} iI_m + \sum_{k=1}^N m_{+,k}(x, \alpha_0) z^{-k} + o\big(|z|^{-N}\big), \quad N \in \mathbb{N}. \tag{4.4}$$

The expansion (4.4) is uniform with respect to $\arg(z)$ for $|z| \to \infty$ in C_ε and uniform in x as long as x varies in compact subintervals of $[x_0, \infty)$ intersected with the right Lebesgue set of $B^{(N-1)}$. The expansion coefficients $m_{+,k}(x, \alpha_0)$ can be

recursively computed from

$$
\begin{aligned}
m_{+,1}(x,\alpha_0) &= -\frac{1}{2}\big(B_{1,2}(x)+B_{2,1}(x)\big)+\frac{i}{2}\big(B_{1,1}(x)-B_{2,2}(x)\big),\\
m_{+,k+1}(x,\alpha_0) &= \frac{i}{2}\bigg(m'_{+,k}(x,\alpha_0)+\sum_{\ell=1}^{k} m_{+,\ell}(x,\alpha_0)m_{+,k+1-\ell}(x,\alpha_0)\\
&\qquad + \sum_{\ell=0}^{k} m_{+,\ell}(x,\alpha_0)B_{2,2}(x)m_{+,k-\ell}(x,\alpha_0)\\
&\qquad + B_{1,2}(x)m_{+,k}(x,\alpha_0)+m_{+,k}(x,\alpha_0)B_{2,1}(x)\bigg),\\
&\qquad\qquad 1\le k\le N-1.
\end{aligned} \tag{4.5}
$$

Proof. In the following let $z\in\mathbb{C}_+$, and $x\ge x_0$. The existence of an expansion of the type (4.4) is shown as follows. First one considers a matrix Volterra integral equation of the type

$$
\widetilde U_+(z,x,\alpha_0)=\begin{pmatrix} I_m\\ iI_m\end{pmatrix}\exp(iz(x-x_0))+\int_x^\infty dx'\,K(z,x,x')JB(x')\widetilde U_+(z,x',\alpha_0), \tag{4.6}
$$

where

$$
\widetilde U_+(z,x,\alpha_0)=\begin{pmatrix}\widetilde u_{+,1}(z,x,\alpha_0)\\ \widetilde u_{+,2}(z,x,\alpha_0)\end{pmatrix}\in L^2([x_0,\infty))^{2m\times m}, \tag{4.7}
$$

and K abbreviates the $2m\times 2m$ Volterra Green's kernel

$$
K(z,x,x')=\begin{pmatrix}\cos(z(x-x'))I_m & \sin(z(x-x'))I_m\\ -\sin(z(x-x'))I_m & \cos(z(x-x'))I_m\end{pmatrix}. \tag{4.8}
$$

Clearly, $\widetilde U_+(z,\cdot,\alpha_0)$ solves the Dirac-type system (2.2) and (2.4). In addition, it satisfies $\widetilde U_+(z,\cdot,\alpha_0)\in L^2([x_0,\infty))^{2m\times 2m}$. Thus, up to normalization, $\widetilde U_+(z,\cdot,\alpha_0)$ represents the Weyl solution associated with B on the half-line $[x_0,\infty)$. Next, introducing

$$
\widetilde V_+(z,x,\alpha_0)=\begin{pmatrix}\widetilde v_{+,1}(z,x,\alpha_0)\\ \widetilde v_{+,2}(z,x,\alpha_0)\end{pmatrix}=\widetilde U_+(z,x,\alpha_0)\exp(-iz(x-x_0)), \tag{4.9}
$$

one rewrites (4.6) in the form

$$
\widetilde V_+(z,x,\alpha_0)=\begin{pmatrix} I_m\\ iI_m\end{pmatrix}+\int_x^{y_0} dx'\,\widetilde K(z,x,x')JB(x')\widetilde V_+(z,x',\alpha_0), \tag{4.10}
$$

where

$$
\widetilde K(z,x,x')=\frac{1}{2}\begin{pmatrix}(1+\exp(2iz(x'-x)))I_m & -i(1-\exp(2iz(x'-x)))I_m\\ i(1-\exp(2iz(x'-x)))I_m & (1+\exp(2iz(x'-x)))I_m\end{pmatrix}. \tag{4.11}
$$

Thus, one infers,

$$
M_+(z,x,\alpha_0)=\widetilde u_{+,2}(z,x,\alpha_0)\widetilde u_{+,1}(z,x,\alpha_0)^{-1}=\widetilde v_{+,2}(z,x,\alpha_0)\widetilde v_{+,1}(z,x,\alpha_0)^{-1}. \tag{4.12}
$$

Introducing

$$
R=\begin{pmatrix} C_1 & -iC_2\\ iC_1 & C_2\end{pmatrix},\quad S=\begin{pmatrix} D_1 & iD_2\\ -iD_1 & D_2\end{pmatrix}, \tag{4.13}
$$

where

$$C_1 = -B_{1,2}^* - iB_{1,1}, \quad C_2 = B_{1,2} - iB_{2,2}, \tag{4.14}$$
$$D_1 = -B_{1,2}^* + iB_{1,1}, \quad D_2 = B_{1,2} + iB_{2,2}, \tag{4.15}$$

(4.10) results in

$$\widetilde{V}_+(z,x,\alpha_0) = \begin{pmatrix} I_m \\ iI_m \end{pmatrix} + \int_x^{y_0} dx' \,\big(R(x') + S(x')\exp(2iz(x'-x))\big)\widetilde{V}_+(z,x',\alpha_0) \tag{4.16}$$

$$\begin{aligned} &= \bigg(I_{2m} + \sum_{k=1}^{\infty} 2^{-k} \int_x^{y_0} dx_1 \,\big(R(x_1) + S(x_1)e^{2iz(x_1-x)}\big) \times \\ &\qquad \times \int_{x_1}^{y_0} dx_2 \,\big(R(x_2) + S(x_2)e^{2iz(x_2-x_1)}\big) \dots \\ &\qquad \cdots \int_{x_{k-1}}^{y_0} dx_k \,\big(R(x_k) + S(x_k)e^{2iz(x_k-x_{k-1})}\big)\bigg) \begin{pmatrix} I_m \\ iI_m \end{pmatrix}. \end{aligned} \tag{4.17}$$

This yields

$$\|\widetilde{v}_{+,j}(z,x,\alpha_0)\| \leq C_j, \quad z \in \mathbb{C}_+, \ \mathrm{Im}(z) > 0, \ x \geq x_0, \ j = 1,2 \tag{4.18}$$

for some $C_j > 0$, $j = 1,2$, depending on $\|B\|_1$. Integrating by parts in (4.17), repeatedly applying (4.1) and (4.3) to $q(x) = (S(x))_{j,k}$ for all $1 \leq j,k \leq 2m$ then results in the existence of an asymptotic expansion for $\widetilde{V}_+(z,x,\alpha_0)$ of the type

$$\widetilde{V}_+(z,x,\alpha_0) = \begin{pmatrix} \widetilde{v}_{+,1}(z,x,\alpha_0) \\ \widetilde{v}_{+,2}(z,x,\alpha_0) \end{pmatrix} = \sum_{k=0}^{N} \widetilde{V}_{+,k}(x,\alpha_0)\, z^{-k} + o\big(|z|^{-N}\big). \tag{4.19}$$

Inserting the expansions for $\widetilde{v}_{+,2}(z,x,\alpha_0)$ and $\widetilde{v}_{+,1}(z,x,\alpha_0)^{-1}$ into (4.12) (using a geometric series expansion for $\widetilde{v}_{+,1}(z,x,\alpha_0)^{-1}$) then yields the existence of an expansion of the type (4.4) for $M_+(z,x,\alpha_0)$. The actual expansion coefficients and the associated recursion relation (4.5) then follow upon inserting expansion (4.4) into the Riccati-type equation (3.16a). The stated uniformity assertions concerning the asymptotic expansion (4.4) then follow from iterating the system of Volterra integral integral equations (4.10). □

Remark 4.2. The analogous solution $\widetilde{U}_-(z,\cdot,\alpha_0)$ of the Dirac-type operator (2.75) on the interval $(-\infty, x_0]$ satisfies

$$\begin{aligned} \widetilde{U}_-(z,x,\alpha_0) &= \begin{pmatrix} I_m \\ -iI_m \end{pmatrix} \exp(-iz(x-x_0)) \\ &\quad - \int_{-\infty}^{x} dx' \, K(z,x,x') J\widetilde{B}(x')\widetilde{U}_-(z,x',\alpha_0), \end{aligned} \tag{4.20}$$

with integral kernel K given by (4.8). (Again $\widetilde{U}_-$ coincides with the Weyl solution U_- up to normalization.) A closer look at the system of Volterra integral equations (4.6), (4.16), (4.17), and similarly in connection with (4.20), then reveals that

$\widetilde{U}_\pm(z,\cdot,\alpha_0)$ have the asymptotic behavior

$$\widetilde{U}_\pm(z,x,\alpha_0) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \left(\sum_{k=0}^{N} \begin{pmatrix} \widetilde{v}_{\pm,k,1}(x,\alpha_0) \\ \widetilde{v}_{\pm,k,2}(x,\alpha_0)\end{pmatrix} z^{-k} + o\big(|z|^{-N}\big)\right)\exp(\pm iz(x-x_0)), \tag{4.21}$$

with leading asymptotics determined as follows.

$$\begin{aligned} \widetilde{v}_{\pm,0,1}(x,\alpha_0) &= I_m + \widetilde{w}_{\pm,0,1}(x,\alpha_0), \\ \widetilde{v}_{\pm,0,2}(x,\alpha_0) &= \pm i\big(I_m + \widetilde{w}_{\pm,0,1}(x,\alpha_0)\big), \end{aligned} \tag{4.22}$$

where $\widetilde{w}_{\pm,0,1}(x,\alpha_0)$ satisfies

$$\widetilde{w}'_{\pm,0,1}(x,\alpha_0) = \frac{1}{2}\big[\widetilde{B}_{2,1}(x) - \widetilde{B}_{1,2}(x) \pm i\widetilde{B}_{2,2}(x) \pm i\widetilde{B}_{1,1}(x)\big]\big(I_m + \widetilde{w}_{\pm,0,1}(x,\alpha_0)\big), \tag{4.23}$$

and

$$\lim_{x\to\pm\infty} \widetilde{w}_{\pm,0,1}(x,\alpha_0) = 0 \tag{4.24}$$

(in fact, $\widetilde{v}_{\pm,0,1}(\cdot,\alpha_0) = I_m$, $\widetilde{v}_{\pm,0,2}(\cdot,\alpha_0) = \pm iI_m$, and $\widetilde{v}_{\pm,k,j}(\cdot,\alpha_0) = 0$, $j = 1,2$, $1 \leq k \leq N$ outside the support of $\widetilde{B}$). In particular,

$$\widetilde{w}_{\pm,0,1}(x,\alpha_0) = 0 \tag{4.25}$$

and hence

$$\widetilde{U}_\pm(z,x,\alpha_0) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \left(\begin{pmatrix} I_m \\ \pm iI_m\end{pmatrix} + o(1)\right)\exp(\pm iz(x-x_0)), \tag{4.26}$$

if and only if $\widetilde{B}$ is in the normal form

$$\widetilde{B}(x) = \begin{pmatrix} \widetilde{B}_{1,1}(x) & \widetilde{B}_{1,2}(x) \\ \widetilde{B}_{1,2}(x) & -\widetilde{B}_{1,1}(x)\end{pmatrix}, \quad \widetilde{B}^*_{1,1}(x) = \widetilde{B}_{1,1}(x),\ \widetilde{B}^*_{1,2}(x) = \widetilde{B}_{1,2}(x) \text{ a.e.} \tag{4.27}$$

For more details we refer to Lemma 5.1.

Next we recall an elementary result on finite-dimensional evolution equations essentially taken from [98] (cf. also [20, Lemma 4.2]).

Lemma 4.3. ([98].) *Let $\Gamma_j \in L^1_{\text{loc}}(\mathbb{R})^{m\times m}$, $j = 1,2$. Then any $m\times m$ matrix-valued solution X of*

$$X'(x) = \Gamma_1(x)X(x) + X(x)\Gamma_2(x) \text{ for a.e. } x\in\mathbb{R}, \tag{4.28}$$

is of the type

$$X(x) = Y(x)CZ(x), \tag{4.29}$$

where C is a constant $m\times m$ matrix and Y is a fundamental system of solutions of

$$\Psi'(x) = \Gamma_1(x)\Psi(x) \tag{4.30}$$

and Z is a fundamental system of solutions of

$$\Phi'(x) = \Phi(x)\Gamma_2(x). \tag{4.31}$$

The next result provides the proper extension of Lemma 4.3 in [20] in the context of matrix-valued Schrödinger operators (which in turn extended Proposition 2.1 in the scalar context in [49] to the matrix-valued case) to the Dirac-type case under consideration.

Lemma 4.4. *Fix $x_0, y_0 \in \mathbb{R}$ with $y_0 > x_0$. Suppose $A_j = I_{2m}$, $B_j \in L^1([x_0, x_0 + R])^{2m\times 2m}$ for all $R > 0$, $B_j = B_j^*$ a.e. on $[x_0, \infty)$, $j = 1, 2$, and $B_1 = B_2$ a.e. on $[x_0, y_0]$. Let $\alpha_0 = (I_m\ 0) \in \mathbb{C}^{m\times 2m}$ and denote by $M_{j,+}(z, x, \alpha_0)$, $x \geq x_0$, the unique Weyl-Titchmarsh matrix corresponding to the half-line Dirac operators $D_{+,j}(\alpha_0)$, $j = 1, 2$, in (2.88). Then,*

$$\begin{aligned}
&[M'_{1,+}(z, x, \alpha_0) - M'_{2,+}(z, x, \alpha_0)] \\
&= -(z/2)[M_{1,+}(z, x, \alpha_0) + M_{2,+}(z, x, \alpha_0)][M_{1,+}(z, x, \alpha_0) - M_{2,+}(z, x, \alpha_0)] \\
&\quad - (z/2)[M_{1,+}(z, x, \alpha_0) - M_{2,+}(z, x, \alpha_0)][M_{1,+}(z, x, \alpha_0) + M_{2,+}(z, x, \alpha_0)] \\
&\quad - [M_{1,+}(z, x, \alpha_0) + M_{2,+}(z, x, \alpha_0)]B_{2,2}(x)[M_{1,+}(z, x, \alpha_0) - M_{2,+}(z, x, \alpha_0)]/2 \\
&\quad - [M_{1,+}(z, x, \alpha_0) - M_{2,+}(z, x, \alpha_0)]B_{2,2}(x)[M_{1,+}(z, x, \alpha_0) + M_{2,+}(z, x, \alpha_0)]/2 \\
&\quad - B_{1,2}(x)[M_{1,+}(z, x, \alpha_0) - M_{2,+}(z, x, \alpha_0)] \\
&\quad - [M_{1,+}(z, x, \alpha_0) - M_{2,+}(z, x, \alpha_0)]B_{2,1}(x) \quad \text{for a.e. } x \in [x_0, y_0],
\end{aligned} \tag{4.32}$$

where we denoted $B_1 = B_2 = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}$ a.e. on (x_0, y_0).

Proof. This is obvious from (3.16a). □

Lemma 4.5. *Fix $x_0, y_0 \in \mathbb{R}$ with $y_0 > x_0$. Suppose $A_j = I_{2m}$, $B_j \in L^1([x_0, x_0 + R])^{2m\times 2m}$ for all $R > 0$, and $B_j = B_j^*$ a.e. on $[x_0, \infty)$, $j = 1, 2$. Let $\alpha_0 = (I_m\ 0) \in \mathbb{C}^{m\times 2m}$ and denote by $M_{j,+}(z, x, \alpha_0)$, $x \geq x_0$, the unique Weyl-Titchmarsh matrix corresponding to the half-line Dirac operators $D_{+,j}(\alpha_0)$, $j = 1, 2$, in (2.88). Define*

$$\begin{aligned}
\Gamma_1(z, x) &= -(z/2)[M_{1,+}(z, x, \alpha_0) + M_{2,+}(z, x, \alpha_0)] \\
&\qquad - (1/2)[M_{1,+}(z, x, \alpha_0) + M_{2,+}(z, x, \alpha_0)]B_{2,2}(x) - B_{1,2}(x),
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
\Gamma_2(z, x) &= -(z/2)[M_{1,+}(z, x, \alpha_0) + M_{2,+}(z, x, \alpha_0)] \\
&\qquad - (1/2)B_{2,2}(x)[M_{1,+}(z, x, \alpha_0) + M_{2,+}(z, x, \alpha_0)] - B_{2,1}(x),
\end{aligned} \tag{4.34}$$

for a.e. $x \in [x_0, y_0]$. In addition, assume $Y_+(z, \cdot)$ and $Z_+(z, \cdot)$ to be fundamental matrix solutions of

$$\Psi'(z, x) = \Gamma_1(z, x)\Psi(z, x) \ \text{and} \ \Phi'(z, x) = \Phi(z, x)\Gamma_2(z, x) \tag{4.35}$$

on $[x_0, y_0]$, respectively, with

$$Y_+(z, y_0) = I_m, \quad Z_+(z, y_0) = I_m. \tag{4.36}$$

Then, as $|z| \to \infty$, $z \in C_\varepsilon$,

$$\|Y_+(z, x_0)\|_{\mathbb{C}^{m\times m}}, \|Z_+(z, x_0)\|_{\mathbb{C}^{m\times m}} \leq \exp(-\mathrm{Im}(z)(y_0 - x_0)(1 + o(1))). \tag{4.37}$$

Proof. Define $\widetilde{\Gamma}_j(z, x)$, $j = 1, 2$, by

$$\widetilde{\Gamma}_j(z, x) = \Gamma_j(z, x) + izI_m, \quad j = 1, 2, \tag{4.38}$$

then

$$\int_{x_0}^{y_0} dx\, \|\widetilde{\Gamma}_j(z, x)\|_{\mathbb{C}^{m\times m}} \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} o(z), \quad j = 1, 2 \tag{4.39}$$

due to the uniform nature of the asymptotic expansion (3.61) for x varying in compact intervals. Next, introduce

$$E_+(z,x,y_0) = I_m \exp(iz(y_0-x)), \quad x \le y_0, \tag{4.40}$$

then

$$Y_+(z,x) = E_+(z,x,y_0) - \int_x^{y_0} dx'\, E_+(z,x,x')\widetilde{\Gamma}_1(z,x')Y_+(z,x'), \tag{4.41}$$

$$Z_+(z,x) = E_+(z,x,y_0) - \int_x^{y_0} dx'\, Z_+(z,x')\widetilde{\Gamma}_2(z,x')E_+(z,x,x'). \tag{4.42}$$

Using

$$\|E_+(z,x_0,y_0)\|_{\mathbb{C}^{m\times m}} \le \exp(-\mathrm{Im}(z)(y_0-x_0)), \tag{4.43}$$

a standard Volterra-type iteration argument in (4.41), (4.42) then yields

$$\|Y_+(z,x_0)\|_{\mathbb{C}^{m\times m}} \le \exp\left(-\mathrm{Im}(z)(y_0-x_0) + \int_{x_0}^{y_0} dx\, \|\widetilde{\Gamma}_1(z,x)\|\right), \tag{4.44}$$

$$\|Z_+(z,x_0)\|_{\mathbb{C}^{m\times m}} \le \exp\left(-\mathrm{Im}(z)(y_0-x_0) + \int_{x_0}^{y_0} dx\, \|\widetilde{\Gamma}_2(z,x)\|\right), \tag{4.45}$$

and hence (4.37). ∎

Theorem 4.6. *Fix $x_0, y_0 \in \mathbb{R}$ with $y_0 > x_0$. Suppose $A_j = I_{2m}$, $B_j \in L^1([x_0, x_0 + R])^{2m\times 2m}$ for all $R > 0$, $B_j = B_j^*$ a.e. on $[x_0,\infty)$, $j = 1,2$, and $B_1 = B_2$ a.e. on $[x_0, y_0]$. Let $\alpha_0 = (I_m\ 0) \in \mathbb{C}^{m\times 2m}$ and denote by $M_{j,+}(z,x,\alpha_0)$, $x \ge x_0$, the unique Weyl-Titchmarsh matrix corresponding to the half-line Dirac operators $D_{+,j}(\alpha_0)$, $j = 1,2$, in* (2.88)*. Then, as $|z| \to \infty$ in C_ε,*

$$\|M_{1,+}(z,x_0,\alpha_0) - M_{2,+}(z,x_0,\alpha_0)\|_{\mathbb{C}^{m\times m}} \le C \exp(-2\mathrm{Im}(z)(y_0-x_0)(1+o(1))) \tag{4.46}$$

for some constant $C > 0$.

Proof. Define for $z \in \mathbb{C}\backslash\mathbb{R}$, $x \in [x_0, y_0]$,

$$X_+(z,x) = M_{1,+}(z,x,\alpha_0) - M_{2,+}(z,x,\alpha_0), \tag{4.47}$$

and for $z \in \mathbb{C}\backslash\mathbb{R}$ and a.e. $x \in [x_0, y_0]$,

$$\begin{aligned} \Gamma_1(z,x) = &-(z/2)[M_{1,+}(z,x_0,\alpha_0) + M_{2,+}(z,x_0,\alpha_0)] \\ &- (1/2)[M_{1,+}(z,x_0,\alpha_0) + M_{2,+}(z,x_0,\alpha_0)]B_{2,2}(x) - B_{1,2}(x), \end{aligned} \tag{4.48}$$

$$\begin{aligned} \Gamma_2(z,x) = &-(z/2)[M_{1,+}(z,x_0,\alpha_0) + M_{2,+}(z,x_0,\alpha_0)] \\ &- (1/2)B_{2,2}(x)[M_{1,+}(z,x_0,\alpha_0) + M_{2,+}(z,x_0,\alpha_0)] - B_{2,1}(x). \end{aligned} \tag{4.49}$$

By Lemma 4.4,

$$X_+' = \Gamma_1 X_+ + X_+ \Gamma_2 \tag{4.50}$$

and hence by Lemma 4.3,

$$X_+(z,x) = Y_+(z,x)X_+(z,x_1)Z_+(z,x), \tag{4.51}$$

where $Y_+(z,x)$ and $Z_+(z,x)$ are fundamental solution matrices of

$$\Psi'(z,x) = \Gamma_1(z,x)\Psi(z,x) \text{ and } \Phi'(z,x) = \Phi(z,x)\Gamma_2(z,x), \tag{4.52}$$

respectively, with

$$Y_+(z,y_0) = I_m, \quad Z_+(z,y_0) = I_m. \tag{4.53}$$

By Lemma 4.5,

$$\|Y_+(z,x_0)\|_{\mathbb{C}^{m\times m}}, \|Z_+(z,x_0)\|_{\mathbb{C}^{m\times m}} \leq \exp(-\mathrm{Im}(z)(y_0-x_0))(1+o(1))) \tag{4.54}$$

as $|z|\to\infty$, $z\in C_\varepsilon$. Thus, as $|z|\to\infty$, $z\in C_\varepsilon$,

$$\begin{aligned}\|X_+(z,x_0)\|_{\mathbb{C}^{m\times m}} &\leq \|X_+(z,y_0)\|_{\mathbb{C}^{m\times m}}\, \|Y_+(z,x_0)\|_{\mathbb{C}^{m\times m}}\, \|Z_+(z,x_0)\|_{\mathbb{C}^{m\times m}} \\ &\leq C\exp(-2\mathrm{Im}(z)(y_0-x_0)(1+o(1)))\end{aligned} \tag{4.55}$$

for some constant $C>0$ by (3.61), (4.51), and (4.54). □

Given these preparations we can now drop the compact support assumption on B in Lemma 4.1 and hence arrive at one of the principal results of this paper.

Theorem 4.7. *Fix $x_0,y_0\in\mathbb{R}$ with $y_0>x_0$ and suppose $A=I_{2m}$, $B\in L^1([x_0,x_0+R])^{2m\times 2m}$ for all $R>0$, and $B=B^*$ a.e. on (x_0,∞). In addition, assume that for some $N\in\mathbb{N}$, $B^{(N-1)}\in L^1([x_0,c])^{2m\times 2m}$ for all $c>x_0$, that x_0 is a right Lebesgue point of $B^{(N-1)}$, and that*

$$\begin{aligned}&\operatorname*{ess\,sup}_{y\in[x_0,y_0]}\bigg\|\int_y^{y_0}dx'\,B^{(N-1)}(x')\exp(2iz(x'-y))+\frac{1}{2iz}B^{(N-1)}(y)\bigg\|_{\mathbb{C}^{2m\times 2m}}\\ &\underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} o\big(|z|^{-1}\big).\end{aligned} \tag{4.56}$$

If $N=1$, suppose in addition $B_{k,k'}B_{\ell,\ell'}\in L^1([x_0,y_0])^{m\times m}$ for all $k,k',\ell,\ell'\in\{1,2\}$. Let $\alpha_0=(I_m\ 0)\in\mathbb{C}^{m\times 2m}$ and denote by $M_+(z,x_0,\alpha_0)$ the unique element of the limit disk $\mathcal{D}_+(z,x_0,\alpha_0)$ for the half-line Dirac operator $D_+(\alpha_0)$ in (2.88). Then, as $|z|\to\infty$ in C_ε, $M_+(z,x_0,\alpha_0)$ has an asymptotic expansion of the form

$$M_+(z,x_0,\alpha_0) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} iI_m+\sum_{k=1}^N m_{+,k}(x_0,\alpha_0)z^{-k}+o\big(|z|^{-N}\big), \quad N\in\mathbb{N}. \tag{4.57}$$

The expansion (4.57) is uniform with respect to $\arg(z)$ for $|z|\to\infty$ in C_ε. The expansion coefficients $m_{+,k}(x_0,\alpha_0)$ can be recursively computed from (4.5).

Proof. Define

$$\widetilde{B}(x)=\begin{cases}B(x) & \text{for } x\in[x_0,y_0],\ x_0<y_0\\ 0 & \text{otherwise}\end{cases} \tag{4.58}$$

and apply Theorem 4.6 with $B_1=B$, $B_2=\widetilde{B}$. Then (in obvious notation)

$$\|M_+(z,x_0,\alpha_0)-\widetilde{M}_+(z,x_0,\alpha_0)\|_{\mathbb{C}^{m\times m}} \leq C\exp(-2\mathrm{Im}(z)(y_0-x_0)(1+o(1))) \tag{4.59}$$

as $|z|\to\infty$, $z\in C_\varepsilon$, and hence the asymptotic expansion (4.4) for $\widetilde{M}_+(z,x_0,\alpha_0)$ in Lemma 4.1 coincides with that of $M_+(z,x_0,\alpha_0)$. □

In analogy to Theorem 3.11, the asymptotic expansion (4.57) extends to one for $M_+(z,x,\alpha_0)$ valid uniformly with respect to x as long as x varies in compact subintervals of $[x_0,\infty)$ intersected with the right Lebesgue set of $B^{(N-1)}$.

Theorem 4.8. *Fix $x_0\in\mathbb{R}$ and let $x\geq x_0$. Suppose $A=I_{2m}$, $B\in L^1([x_0,x_0+R])^{2m\times 2m}$ for all $R>0$, and $B=B^*$ a.e. on (x_0,∞). In addition, assume that for*

some $N \in \mathbb{N}$, $B^{(N-1)} \in L^1([x_0,c))^{2m\times 2m}$ for all $c > x_0$, that x is a right Lebesgue point of $B^{(N-1)}$, and that for all $R > 0$,

$$\operatorname*{ess\,sup}_{y\in[x_0,x_0+R]} \bigg\| \int_y^{x_0+R} dx'\, B^{(N-1)}(x') \exp(2iz(x'-y)) + \frac{1}{2iz} B^{(N-1)}(y) \bigg\|_{\mathbb{C}^{2m\times 2m}} \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} o\big(|z|^{-1}\big). \tag{4.60}$$

If $N = 1$, suppose in addition $B_{k,k'}B_{\ell,\ell'} \in L^1([x_0,x_0+R])^{m\times m}$ for all $R > 0$ and all $k,k',\ell,\ell' \in \{1,2\}$. Let $\alpha_0 = (I_m\ 0) \in \mathbb{C}^{m\times 2m}$ and denote by $M_+(z,x,\alpha_0)$, $x \geq x_0$, the unique element of the limit disk $\mathcal{D}_+(z,x,\alpha_0)$ for the half-line Dirac operator $D_+(\alpha_0)$ in (2.88). Then, as $|z| \to \infty$ in C_ε, $M_+(z,x,\alpha_0)$ has an asymptotic expansion of the form

$$M_+(z,x,\alpha_0) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} iI_m + \sum_{k=1}^N m_{+,k}(x,\alpha_0) z^{-k} + o\big(|z|^{-N}\big), \quad N \in \mathbb{N}. \tag{4.61}$$

The expansion (4.61) is uniform with respect to $\arg(z)$ for $|z| \to \infty$ in C_ε and uniform in x as long as x varies in compact subsets of $\mathbb{R}$ intersected with the right Lebesgue set of $B^{(N-1)}$. The expansion coefficients $m_{+,k}(x,\alpha_0)$ can be recursively computed from (4.5).

Proof. To see that uniformity holds for this expansion, first recall the role of Theorem 3.11 in providing uniformity in the asymptotic expression (4.39) which then leads to (4.37) holding uniformly with respect to x_0 varying within compact subsets of $\mathbb{R}$ and with respect to $\arg(z)$ for $|z| \to \infty$ in C_ε. This in turn leads to a similar uniformity holding for (4.46) which is the key to (4.57) holding with respect to x_0 varying within compact subsets of $\mathbb{R}$ and with respect to $\arg(z)$ for $|z| \to \infty$ in C_ε. □

Remark 4.9. For simplicity, we focused thus far on the expansion of $M_+(z,x_0,\alpha_0)$ as $|z| \to \infty$. Of course, Theorem 4.8 holds also for $M_-(z,x_0,\alpha_0)$ replacing the hypotheses concerning right Lebesgue points by those of left Lebesgue points, etc. For convenience we just state the corresponding expansion and associated nonlinear recursion formula which covers both cases.

$$M_\pm(z,x,\alpha_0) \underset{\substack{|z|\to\infty\\ z\in C_\varepsilon}}{=} \sum_{k=0}^N m_{\pm,k}(x,\alpha_0) z^{-k} + o\big(|z|^{-N}\big), \quad N \in \mathbb{N}. \tag{4.62}$$

$$\begin{aligned}
m_{\pm,0}(x,\alpha_0) &= \pm i I_m,\\
m_{\pm,1}(x,\alpha_0) &= -\frac{1}{2}\big(B_{1,2}(x) + B_{2,1}(x)\big) \pm \frac{i}{2}\big(B_{1,1}(x) - B_{2,2}(x)\big),\\
m_{\pm,k+1}(x,\alpha_0) &= \pm\frac{i}{2}\bigg(m'_{\pm,k}(x,\alpha_0) + \sum_{\ell=1}^k m_{\pm,\ell}(x,\alpha_0) m_{\pm,k+1-\ell}(x,\alpha_0)\\
&\quad + \sum_{\ell=0}^k m_{\pm,\ell}(x,\alpha_0) B_{2,2}(x) m_{\pm,k-\ell}(x,\alpha_0)\\
&\quad + B_{1,2}(x) m_{\pm,k}(x,\alpha_0) + m_{\pm,k}(x,\alpha_0) B_{2,1}(x)\bigg),
\end{aligned} \tag{4.63}$$

$$1 \leq k \leq N-1.$$

Combining Theorem 4.8 and (2.77) then yields the analogous asymptotic expansion for $M(z,x,\alpha_0)$.

Theorem 4.10. *Assume Hypothesis 2.1 with $A = I_{2m}$, and let $\alpha_0 = (I_m\ 0) \in \mathbb{C}^{m\times 2m}$. Fix $x_0 \in \mathbb{R}$ and let $x \in \mathbb{R}$. Suppose that for some $N \in \mathbb{N}$, $B^{(N-1)} \in L^1_{\text{loc}}(\mathbb{R})^{2m\times 2m}$, that x is a right and a left Lebesgue point of $B^{(N-1)}$, and that for all $R > 0$,*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{y\in[x_0,x_0+R]} \bigg\| \int_y^{x_0+R} dx'\, B^{(N-1)}(x') \exp(2iz(x'-y)) + \frac{1}{2iz} B^{(N-1)}(y) \bigg\|_{\mathbb{C}^{2m\times 2m}} \\
&+ \operatorname*{ess\,sup}_{y\in[x_0-R,x_0]} \bigg\| \int_{x_0-R}^{y} dx'\, B^{(N-1)}(x') \exp(2iz(x'-y)) - \frac{1}{2iz} B^{(N-1)}(y) \bigg\|_{\mathbb{C}^{2m\times 2m}} \\
&\underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} o\big(|z|^{-1}\big).
\end{aligned} \tag{4.64}$$

If $N = 1$, assume in addition $B_{k,k'}B_{\ell,\ell'} \in L^1_{\text{loc}}(\mathbb{R})^{m\times m}$ for all $k,k',\ell,\ell' \in \{1,2\}$. Let $M(z,x,\alpha_0)$ be defined as in (2.76) (see also (2.77)). Then, as $|z| \to \infty$ in C_ε, $M(z,x,\alpha_0)$ has an asymptotic expansion of the form

$$M(z,x,\alpha_0) \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} (i/2)I_{2m} + \sum_{k=1}^{N} M_k(x,\alpha_0) z^{-k} + o\big(|z|^{-N}\big), \quad N \in \mathbb{N}, \tag{4.65}$$

where

$$\begin{aligned}
M_1(x,\alpha_0) = &-\frac{i}{8} \begin{pmatrix} B_{1,1}(x+0) - B_{2,2}(x+0) & B_{1,2}(x+0) + B_{2,1}(x+0) \\ B_{1,2}(x+0) + B_{2,1}(x+0) & B_{2,2}(x+0) - B_{1,1}(x+0) \end{pmatrix} \\
&- \frac{i}{8} \begin{pmatrix} B_{1,1}(x-0) - B_{2,2}(x-0) & B_{1,2}(x-0) + B_{2,1}(x-0) \\ B_{1,2}(x-0) + B_{2,1}(x-0) & B_{2,2}(x-0) - B_{1,1}(x-0) \end{pmatrix}, \ \textit{etc.}
\end{aligned} \tag{4.66}$$

The expansion (4.65) is uniform with respect to $\arg(z)$ for $|z| \to \infty$ in C_ε and uniform in x as long as x varies in compact subsets of $\mathbb{R}$ intersected with the right and left Lebesgue set of $B^{(N-1)}$.
If one merely assumes Hypothesis 2.1 with $A = I_{2m}$, $\alpha_0 = (I_m\ 0)$, and $B \in L^1_{\text{loc}}(\mathbb{R})^{2m\times 2m}$, then

$$M(z,x,\alpha_0) \underset{\substack{|z|\to\infty \\ z\in C_\varepsilon}}{=} (i/2)I_{2m} + o(1). \tag{4.67}$$

Again the asymptotic expansion (4.67) is uniform with respect to $\arg(z)$ for $|z| \to \infty$ in C_ε and uniform in $x \in \mathbb{R}$ as long as x varies in compact intervals.

The higher-order coefficients in (4.65) can be derived upon inserting (4.62) into (3.16a), taking into account (2.77).

Theorems 4.7 and 4.8 (with $N \in \mathbb{N}$) are new even in the scalar case $m = 1$ with respect to the regularity assumptions on B. For previous results in the case $m = 1$ under stronger hypotheses on B we refer to [32], [56], [60], [61], [97]. In particular, [56], [60], and [61] derived alternative high-energy expansions for the Weyl-Titchmarsh m-function in the case $m = 1$.

Throughout this section we fixed α to be $\alpha_0 = (I_m\ 0)$. The case of general $\alpha \in \mathbb{C}^{2m\times m}$ satisfying (2.9) then follows from (2.60).

5. A Local Uniqueness Result

In this section we assume that B is in the normal form given in Theorem 1.1,

$$B(x) = \begin{pmatrix} B_{1,1}(x) & B_{1,2}(x) \\ B_{1,2}(x) & -B_{1,1}(x) \end{pmatrix}, \tag{5.1}$$

with $B_{1,1}$ and $B_{1,2}$ self-adjoint a.e. We prove fundamental new local uniqueness results for B in terms of exponentially small differences of Weyl-Titchmarsh matrices $M_+(z,x,\alpha)$ and $M(z,x,\alpha)$. These results, in turn, yield new global ramifications. We start with an auxiliary result concerning asymptotic expansions.

Lemma 5.1. *Suppose $\alpha = (\alpha_1\ \alpha_2) \in \mathbb{C}^{m\times 2m}$ satisfies* (2.9)*, fix $x_0, y_0 \in \mathbb{R}$ with $y_0 > x_0$, and let $x \geq x_0$. Assume $A = I_{2m}$, $B \in L^1([x_0,\infty))^{2m\times 2m}$,* supp$(B) \subseteq$ *$[x_0, y_0]$, with B in the normal form given in* (5.1) *a.e. on (x_0, y_0). Then, the following asymptotic expansions hold for $\Theta(z,x,x_0,\alpha)$, $\Phi(z,x,x_0,\alpha)$, and $U_+(z,x,x_0,\alpha)$ associated with* (2.2a)*,*

$$\Theta(z,x,x_0,\alpha) \underset{\substack{|z|\to\infty \\ z\in\mathbb{C}_+}}{=} \frac{1}{2}\begin{pmatrix} \alpha_1^* + i\alpha_2^* \\ -i(\alpha_1^* + i\alpha_2^*) \end{pmatrix} \exp(-iz(x-x_0))\big(1+o(1)\big), \quad x > x_0, \tag{5.2}$$

$$\Phi(z,x,x_0,\alpha) \underset{\substack{|z|\to\infty \\ z\in\mathbb{C}_+}}{=} \frac{i}{2}\begin{pmatrix} -\alpha_2^* + i\alpha_1^* \\ -i(-\alpha_2^* + i\alpha_1^*) \end{pmatrix} \exp(-iz(x-x_0))\big(1+o(1)\big), \quad x > x_0, \tag{5.3}$$

$$U_+(z,x,x_0,\alpha) \underset{\substack{|z|\to\infty \\ z\in\mathbb{C}_+}}{=} \begin{pmatrix} \alpha_1^* - i\alpha_2^* \\ i(\alpha_1^* - i\alpha_2^*) \end{pmatrix} \exp(iz(x-x_0))\big(1+o(1)\big), \quad x \geq x_0. \tag{5.4}$$

Next, we introduce the abbreviation

$$C = -B_{1,2} - iB_{1,1}, \quad C^* = -B_{1,2} + iB_{1,1}, \tag{5.5}$$

and suppose in addition that

$$\operatorname*{ess\,sup}_{y\in[x_0,y_0]} \bigg\| \int_y^{y_0} dx'\, B(x') \exp(2iz(x'-y)) + \frac{1}{2iz} B(y) \bigg\|_{\mathbb{C}^{2m\times 2m}} \underset{\substack{|z|\to\infty \\ z\in\rho_+}}{=} o\big(|z|^{-1}\big), \tag{5.6}$$

along a ray $\rho_+ \subset \mathbb{C}_+$, and that

$$B_{1,1}^2,\ B_{1,2}^2,\ B_{1,1}B_{1,2},\ B_{1,2}B_{1,1} \in L^1([x_0,y_0])^{m\times m}. \tag{5.7}$$

Then,

$$\begin{aligned} \Theta(z,x,x_0,\alpha) \underset{\substack{|z|\to\infty \\ z\in\rho_+}}{=} & \Bigg(\frac{1}{2}\begin{pmatrix} \alpha_1^* + i\alpha_2^* \\ -i(\alpha_1^* + i\alpha_2^*) \end{pmatrix} - \frac{i}{4z}\begin{pmatrix} C(x_0)^*(\alpha_1^* - i\alpha_2^*) \\ -iC(x_0)^*(\alpha_1^* - i\alpha_2^*) \end{pmatrix} \\ & - \frac{i}{4z}\begin{pmatrix} C(x)(\alpha_1^* + i\alpha_2^*) \\ iC(x)(\alpha_1^* + i\alpha_2^*) \end{pmatrix} \\ & + \frac{i}{4z}\int_{x_0}^x dx' \begin{pmatrix} C(x')^*C(x')(\alpha_1^* + i\alpha_2^*) \\ -iC(x')^*C(x')(\alpha_1^* + i\alpha_2^*) \end{pmatrix} \Bigg) e^{-iz(x-x_0)}\big(1+o\big(|z|^{-1}\big)\big), \\ & \hspace{8cm} x > x_0, \end{aligned} \tag{5.8}$$

$$\Phi(z,x,x_0,\alpha) \underset{\substack{|z|\to\infty \\ z\in\rho_+}}{=} \Bigg(\frac{i}{2}\begin{pmatrix} -\alpha_2^* + i\alpha_1^* \\ -i(-\alpha_2^* + i\alpha_1^*) \end{pmatrix} - \frac{1}{4z}\begin{pmatrix} C(x_0)^*(-\alpha_2^* - i\alpha_1^*) \\ -iC(x_0)^*(-\alpha_2^* - i\alpha_1^*) \end{pmatrix}$$

$$+\frac{1}{4z}\begin{pmatrix} C(x)(-\alpha_2^*+i\alpha_1^*) \\ iC(x)(-\alpha_2^*+i\alpha_1^*)\end{pmatrix}$$
$$-\frac{1}{4z}\int_{x_0}^{x} dx'\,\begin{pmatrix} C(x')^*C(x')(-\alpha_2^*+i\alpha_1^*) \\ -iC(x')^*C(x')(-\alpha_2^*+i\alpha_1^*)\end{pmatrix}\Bigg)e^{-iz(x-x_0)}\big(1+o\big(|z|^{-1}\big)\big),$$
$$x > x_0, \qquad (5.9)$$

whenever x_0 is a right Lebesgue point of B and x is a left Lebesgue point of B, and

$$U_+(z,x,x_0,\alpha) \underset{\substack{|z|\to\infty\\ z\in\rho_+}}{=} \Bigg(\begin{pmatrix} \alpha_1^*-i\alpha_2^* \\ i(\alpha_1^*-i\alpha_2^*)\end{pmatrix}+\frac{i}{2z}\begin{pmatrix} (C(x)^*-C(x_0)^*)(\alpha_1^*-i\alpha_2^*) \\ -i(C(x)^*+C(x_0)^*)(\alpha_1^*-i\alpha_2^*)\end{pmatrix}$$
$$-\frac{i}{2z}\int_{x_0}^{x} dx'\,\begin{pmatrix} C(x')C(x')^*(\alpha_1^*-i\alpha_2^*) \\ iC(x')C(x')^*(\alpha_1^*-i\alpha_2^*)\end{pmatrix}\Bigg)e^{iz(x-x_0)}\big(1+o\big(|z|^{-1}\big)\big), \quad x\geq x_0, \tag{5.10}$$

whenever x is a right Lebesgue point of B.

Proof. Since x_0 and α are fixed throughout this proof, we will temporarily suppress these variables whenever possible to simplify notations. Introducing

$$\widehat\Theta(z,x)=2\Theta(z,x)\exp(iz(x-x_0)), \tag{5.11}$$

the Volterra integral equation for Θ (cf. (4.8)),

$$\Theta(z,x)=\begin{pmatrix} \alpha_1^*\cos(z(x-x_0))+\alpha_2^*\sin(z(x-x_0)) \\ \alpha_2^*\cos(z(x-x_0))-\alpha_1^*\sin(z(x-x_0))\end{pmatrix} - \int_{x_0}^{x} dx'\,K(z,x,x')JB(x')\Theta(z,x'), \tag{5.12}$$

can be rewritten in terms of that of $\widehat\Theta$ in the form

$$\widehat\Theta(z,x)=\begin{pmatrix} \alpha_1^*+i\alpha_2^* \\ -i(\alpha_1^*+i\alpha_2^*)\end{pmatrix}+\begin{pmatrix} \alpha_1^*-i\alpha_2^* \\ i(\alpha_1^*-i\alpha_2^*)\end{pmatrix}\exp(2iz(x-x_0)) - \frac{1}{2}\int_{x_0}^{x} dx'\big(R(x')\exp(2iz(x-x'))+S(x')\big)\widehat\Theta(z,x'), \tag{5.13}$$

where we abbreviated

$$R=\begin{pmatrix} C & iC \\ iC & -C\end{pmatrix}, \quad S=\begin{pmatrix} C^* & -iC^* \\ -iC^* & -C^*\end{pmatrix}. \tag{5.14}$$

Using the elementary algebraic facts

$$R\begin{pmatrix} a \\ ia\end{pmatrix}=0, \quad R\begin{pmatrix} b \\ -ib\end{pmatrix}=2\begin{pmatrix} Cb \\ iCb\end{pmatrix}, \quad S\begin{pmatrix} a \\ ia\end{pmatrix}=2\begin{pmatrix} C^*a \\ -iC^*a\end{pmatrix}, \quad S\begin{pmatrix} b \\ -ib\end{pmatrix}=0 \tag{5.15}$$

for any $a,b\in\mathbb{C}^{m\times m}$, iterating (5.13) yields

$$\widehat\Theta(z,x)=\begin{pmatrix} \alpha_1^*+i\alpha_2^* \\ -i(\alpha_1^*+i\alpha_2^*)\end{pmatrix}+\begin{pmatrix} \alpha_1^*-i\alpha_2^* \\ i(\alpha_1^*+i\alpha_2^*)\end{pmatrix}e^{2iz(x-x_0)} + \sum_{m=1}^{\infty}(-2)^{-m}\int_{x_0}^{x} d\xi_1\,\big(R(\xi_1)e^{2iz(x-\xi_1)}+S(\xi_1)\big)\times \times\int_{x_0}^{\xi_1} d\xi_2\,\big(R(\xi_2)e^{2iz(\xi_1-\xi_2)}+S(\xi_2)\big)\dots \tag{5.16}$$

$$\cdots \int_{x_0}^{\xi_{m-2}} d\xi_{m-1} \big(R(\xi_{m-1}) e^{2iz(\xi_{m-2}-\xi_{m-1})} + S(\xi_{m-1})\big) \times$$
$$\times \int_{x_0}^{\xi_{m-1}} d\xi_m \Bigg(R(\xi_m) \begin{pmatrix} \alpha_1^* - i\alpha_2^* \\ -i(\alpha_1^* - i\alpha_2^*) \end{pmatrix} e^{2iz(\xi_{m-1}-\xi_m)}$$
$$+ S(\xi_m) \begin{pmatrix} \alpha_1^* + i\alpha_2^* \\ i(\alpha_1^* + i\alpha_2^*) \end{pmatrix} e^{2iz(\xi_m - x_0)}\Bigg).$$

Applying the Riemann-Lebesgue lemma to (5.16) then proves (5.2) assuming $B \in L^1([x_0,\infty))^{2m\times 2m}$, only. Assuming also (5.6) and (5.7) one can compute the next term in the asymptotic expansion (5.2) and then obtains (5.8) using (5.16) and the finite-interval variant of (4.1), whenever x_0 is a right Lebesgue point of B and x is a left Lebesgue point of B.

Exactly the same arguments apply to Φ. Introducing

$$\widehat{\Phi}(z,x) = 2\Phi(z,x)\exp(iz(x-x_0)), \tag{5.17}$$

the Volterra integral equation for Φ,

$$\begin{aligned} \Phi(z,x) = &\begin{pmatrix} -\alpha_2^* \cos(z(x-x_0)) + \alpha_1^* \sin(iz(x-x_0)) \\ \alpha_1^* \cos(iz(x-x_0)) + \alpha_2^* \sin(z(x-x_0)) \end{pmatrix} \\ &- \int_{x_0}^{x} dx'\, K(z,x,x')JB(x')\Phi_j(z,x'), \end{aligned} \tag{5.18}$$

can be rewritten in terms of that of $\widehat{\Phi}$ in the form

$$\begin{aligned} \widehat{\Phi}(z,x) = &\, i \begin{pmatrix} -\alpha_2^* + i\alpha_1^* \\ -i(-\alpha_2^* + i\alpha_1^*) \end{pmatrix} - i \begin{pmatrix} -\alpha_2^* - i\alpha_1^* \\ i(-\alpha_2^* - i\alpha_1^*) \end{pmatrix} \exp(2iz(x-x_0)) \\ &- \frac{1}{2}\int_{x_0}^{x} \big(R(x')\exp(2iz(x-x')) + S(x')\big)\widehat{\Phi}(z,x'). \end{aligned} \tag{5.19}$$

Iterating (5.19), taking into account (5.15), yields

$$\begin{aligned} \widehat{\Phi}(z,x) = &\, i \begin{pmatrix} -\alpha_2^* + i\alpha_1^* \\ -i(-\alpha_2^* + i\alpha_1^*) \end{pmatrix} - i \begin{pmatrix} -\alpha_2^* - i\alpha_1^* \\ i(-\alpha_2^* - i\alpha_1^*) \end{pmatrix} e^{2iz(x-x_0)} \\ &+ \sum_{m=1}^{\infty} (-2)^{-m} \int_{x_0}^{x} d\xi_1 \big(R(\xi_1)e^{2iz(x-\xi_1)} + S(\xi_1)\big) \times \\ &\times \int_{x_0}^{\xi_1} d\xi_2 \big(R(\xi_2)e^{2iz(\xi_1-\xi_2)} + S(\xi_2)\big) \dots \\ &\cdots \int_{x_0}^{\xi_{m-2}} d\xi_{m-1} \big(R(\xi_{m-1})e^{2iz(\xi_{m-2}-\xi_{m-1})} + S(\xi_{m-1})\big) \times \\ &\times \int_{x_0}^{\xi_{m-1}} d\xi_m \Bigg(iR(\xi_m) \begin{pmatrix} -\alpha_2^* + i\alpha_1^* \\ -i(-\alpha_2^* + i\alpha_1^*) \end{pmatrix} e^{2iz(\xi_{m-1}-\xi_m)} \\ &\qquad - iS(\xi_m) \begin{pmatrix} -\alpha_2^* - i\alpha_1^* \\ i(-\alpha_2^* - i\alpha_1^*) \end{pmatrix} e^{2iz(\xi_m - x_0)} \Bigg). \end{aligned} \tag{5.20}$$

Applying the Riemann-Lebesgue lemma to (5.20) the proves (5.3) assuming $B \in L^1([x_0,\infty))^{2m\times 2m}$, only. Assuming also (5.6) and (5.7) one can compute the next term in the asymptotic expansion (5.3) and then obtains (5.9) using (5.20) and the finite-interval variant of (4.1), whenever x_0 is a right Lebesgue point of B and x is

a left Lebesgue point of B.

Finally, we turn to $U_+(z,x)$. Introducing

$$\widetilde{V}_+(z,x) = \widetilde{U}_+(z,x)\exp(-iz(x-x_0)), \tag{5.21}$$

the Volterra integral equation for $\widetilde{U}_+$,

$$\widetilde{U}_+(z,x) = \begin{pmatrix} \alpha_1^* - i\alpha_2^* \\ i(\alpha_1^* - i\alpha_2^*) \end{pmatrix} \exp(iz(x-x_0)) + \int_x^\infty dx'\, K(z,x,x')JB(x')\widetilde{U}_+(z,x'), \tag{5.22}$$

can be rewritten in terms of that of $\widetilde{V}_+$ in the form

$$\widetilde{V}_+(z,x) = \begin{pmatrix} \alpha_1^* - i\alpha_2^* \\ i(\alpha_1^* - i\alpha_2^*) \end{pmatrix} + \frac{1}{2}\int_x^{y_0} dx' \left(R(x') + S(x')\exp(2iz(x'-x))\right)\widetilde{V}_+(z,x'). \tag{5.23}$$

Iterating (5.23), taking into account (5.15), yields

$$\begin{aligned} \widetilde{V}_+(z,x) = \begin{pmatrix} \alpha_1^* - i\alpha_2^* \\ i(\alpha_1^* - i\alpha_2^*) \end{pmatrix} &+ \sum_{k=1}^\infty 2^{-2k} \int_x^{y_0} d\xi_1\, R(\xi_1) \int_{\xi_1}^{y_0} d\xi_2\, S(\xi_2) e^{2iz(\xi_2 - \xi_1)} \times \\ &\times \int_{\xi_2}^{y_0} d\xi_3\, R(\xi_3) \cdots \int_{\xi_{2k-2}}^{y_0} d\xi_{2k-1}\, R(\xi_{2k-1}) \times \\ &\times \int_{\xi_{2k-1}}^{y_0} d\xi_{2k}\, S(\xi_m) \begin{pmatrix} \alpha_1^* - i\alpha_2^* \\ i(\alpha_1^* - i\alpha_2^*) \end{pmatrix} e^{iz(\xi_{2k} - \xi_{2k-1})} \\ &+ \sum_{\ell=0}^\infty 2^{-2\ell+1} \int_x^{y_0} d\xi_1\, S(\xi_1) e^{2iz(\xi_1 - x)} \int_{\xi_1}^{y_0} d\xi_2\, R(\xi_2) \times \\ &\times \int_{\xi_2}^{y_0} d\xi_3\, S(\xi_3) e^{2iz(\xi_3 - \xi_2)} \cdots \int_{\xi_{2\ell-1}}^{y_0} d\xi_{2\ell}\, R(\xi_{2\ell}) \times \\ &\times \int_{\xi_{2\ell}}^{y_0} d\xi_{2\ell+1}\, S(\xi_{2\ell+1}) \begin{pmatrix} \alpha_1^* - i\alpha_2^* \\ i(\alpha_1^* - i\alpha_2^*) \end{pmatrix} e^{iz(\xi_{2\ell+1} - \xi_{2\ell})}. \end{aligned} \tag{5.24}$$

Next, we take into account the different normalizations of U_+ and $\widetilde{U}_+$. Using $U_+(z,x_0) = [I_m\ M_+(z,x_0)^t]^t$ (cf., (2.69) and $\Psi(z,x_0,x_0,\alpha_0) = I_{2m}$), one readily verifies the relationship

$$u_{+,1}(z,x) = \widetilde{u}_{+,1}(z,x)\widetilde{u}_{+,1}(z,x_0)^{-1}, \quad u_{+,2}(z,x) = \widetilde{u}_{+,2}(z,x)\widetilde{u}_{+,1}(z,x_0)^{-1}. \tag{5.25}$$

Thus, applying the Riemann-Lebesgue lemma to (5.24) then proves (5.4) (in agreement with (4.26)), assuming $B \in L^1([x_0,\infty))^{2m\times 2m}$, only. Assuming also (5.6) and (5.7) one can compute the next term in the asymptotic expansion (5.4) and then obtains (5.10) using (5.24) and (4.1), whenever x is a right Lebesgue point of B. $\square$

In the special case $m = 1$ (and for $\alpha = (1\ 0)$), the expansion (5.10) was stated in [54].

Next, we note an elementary result concerning the boundary data independence of exponentially close Weyl-Titchmarsh matrices.

Lemma 5.2. *Fix $x_0 \in \mathbb{R}$ and suppose $A_j = I_{2m}$, $B_j = B_j^* \in L^1([x_0,x_0+R])^{2m\times 2m}$ for all $R > 0$. Denote by $M_{+,j}(z,x,\alpha)$, $x \geq x_0$, the unique Weyl-Titchmarsh*

matrices corresponding to the half-line Dirac-type operators $D_{+,j}(\alpha)$, $j=1,2$, *in* (2.88)*. Fix an* $\hat\alpha \in \mathbb{C}^{m\times 2m}$ *satisfying* (2.9) *and assume that for all* $\varepsilon>0$,

$$\|M_{+,1}(z,x_0,\hat\alpha)-M_{+,2}(z,x_0,\hat\alpha)\|_{\mathbb{C}^{m\times m}} \underset{\substack{|z|\to\infty\\ z\in\rho_+}}{=} O\big(e^{-2\mathrm{Im}(z)(a-\varepsilon)}\big) \tag{5.26}$$

along some ray $\rho_+\subset\mathbb{C}_+$*. Then, for all* $\alpha\in\mathbb{C}^{m\times 2m}$ *satisfying* (2.9) *and for all* $\varepsilon>0$,

$$\|M_{+,1}(z,x_0,\alpha)-M_{+,2}(z,x_0,\alpha)\|_{\mathbb{C}^{m\times m}} \underset{\substack{|z|\to\infty\\ z\in\rho_+}}{=} O\big(e^{-2\mathrm{Im}(z)(a-\varepsilon)}\big) \tag{5.27}$$

along the ray ρ_+.

Proof. Using (2.57) and (2.60) one estimates

$$\begin{aligned}
&\|M_{+,1}(z,x_0,\alpha)-M_{+,2}(z,x_0,\alpha)\|_{\mathbb{C}^{m\times m}}\\
&=\|M_{+,1}(\bar z,x_0,\alpha)^*-M_{+,2}(z,x_0,\alpha)\|_{\mathbb{C}^{m\times m}}\\
&\le \|[\hat\alpha\alpha^*-M_{+,1}(\bar z,x_0,\hat\alpha)^*\hat\alpha J\alpha^*]^{-1}\|_{\mathbb{C}^{m\times m}}\times\\
&\quad\times\|M_{+,1}(z,x_0,\hat\alpha)-M_{+,2}(z,x_0,\hat\alpha)\|_{\mathbb{C}^{m\times m}}\times\\
&\quad\times\|[\alpha\hat\alpha^*+\alpha J\hat\alpha^*M_{+,2}(z,x_0,\hat\alpha)]^{-1}\|_{\mathbb{C}^{m\times m}},
\end{aligned} \tag{5.28}$$

since by (2.13)

$$\hat\alpha J\alpha^*\alpha\hat\alpha^*+\hat\alpha\alpha^*\alpha J\hat\alpha^*=0,\quad \hat\alpha\alpha^*\alpha\hat\alpha^*-\hat\alpha J\alpha^*\alpha J\hat\alpha^*=I_m. \tag{5.29}$$

Moreover, since

$$[\hat\alpha\alpha^*-i\hat\alpha J\alpha^*][\hat\alpha\alpha^*-i\hat\alpha J\alpha^*]^*=I_m, \tag{5.30}$$

by (5.29), one infers (5.27) from (5.28) and $M_{+,j}(z,x_0,\alpha)=iI_m+o(1)$ as $|z|\to\infty$, $z\in\mathbb{C}_+$, $j=1,2$ (cf. (3.1)). ☐

Our principal new local uniqueness result for Dirac-type operators in terms of Weyl-Titchmarsh matrices then reads as follows.

Theorem 5.3. *Fix* $x_0\in\mathbb{R}$ *and suppose* $A_j=I_{2m}$, $B_j\in L^1([x_0,x_0+R])^{2m\times 2m}$ *for all* $R>0$*. Suppose also that* B_j *is in the normal form given in* (5.1) *a.e. on* (x_0,∞), $j=1,2$*. Denote by* $M_{j,+}(z,x,\alpha)$, $x\ge x_0$*, the unique Weyl-Titchmarsh matrices corresponding to the half-line Dirac-type operators* $D_{+,j}(\alpha)$, $j=1,2$, *in* (2.88)*. Then,*

$$\text{if for some } a>0,\ B_1(x)=B_2(x) \text{ for a.e. } x\in(x_0,x_0+a), \tag{5.31}$$

one obtains

$$\|M_{1,+}(z,x_0,\alpha)-M_{2,+}(z,x_0,\alpha)\|_{\mathbb{C}^{m\times m}} \underset{\substack{|z|\to\infty\\ z\in\rho_+}}{=} O\big(e^{-2\mathrm{Im}(z)a}\big) \tag{5.32}$$

along any ray $\rho_+\subset\mathbb{C}_+$ *with* $0<\arg(z)<\pi$ *and for all* $\alpha\in\mathbb{C}^{m\times 2m}$ *satisfying* (2.9)*. Conversely, fix an* $\hat\alpha\in\mathbb{C}^{m\times 2m}$ *satisfying* (2.9) *and if* $m>1$*, assume in addition that* $B_j\in L^\infty([x_0,x_0+a])^{2m\times 2m}$, $j=1,2$*. Moreover, suppose that for all* $\varepsilon>0$,

$$\|M_{1,+}(z,x_0,\hat\alpha_1)-M_{2,+}(z,x_0,\hat\alpha_1)\|_{\mathbb{C}^{m\times m}} \underset{\substack{|z|\to\infty\\ z\in\rho_{+,\ell}}}{=} O\big(e^{-2\mathrm{Im}(z)(a-\varepsilon)}\big),\quad \ell=1,2, \tag{5.33}$$

along a ray $\rho_{+,1} \subset \mathbb{C}_+$ *with* $0 < \arg(z) < \pi/2$ *and along a ray* $\rho_{+,2} \subset \mathbb{C}_+$ *with* $\pi/2 < \arg(z) < \pi$. *Then*

$$B_1(x) = B_2(x) \text{ for a.e. } x \in [x_0, x_0 + a]. \tag{5.34}$$

Proof. Since (5.32) follows from Theorem 4.6 and Lemma 5.2, it suffices to focus on the proof of (5.34). Moreover, applying Theorem 4.6, we may without loss of generality assume for the rest of the proof that

$$\operatorname{supp}(B_j) \subseteq [x_0, x_0 + a], \quad j = 1, 2. \tag{5.35}$$

In the following, we will adapt the principal ingredients of a recent proof of the local Borg-Marchenko uniqueness theorem for scalar Schrödinger operators (i.e., for $m = 1$) by Bennewitz [13], to the current Dirac-type situation. First we recall that by Lemma 5.2, (5.33) holds along the rays $\rho_{+,j}$, $j = 1, 2$ for all $\alpha = (\alpha_1\ \alpha_2) \in \mathbb{C}^{m\times 2m}$ satisfying (2.9). To simplify notations in the following we will again suppress x_0 and α whenever possible and hence abbreviate, $\Theta(z, x, x_0, \alpha)$, $\Phi(z, x, x_0, \alpha)$, and $U_{j,+}(z, x, x_0, \alpha)$ by $\Theta(z, x)$, $\Phi(z, x)$, and $U_{j,+}(z, x)$, respectively. Next, denoting in obvious notation by

$$\Theta_j(z,x) = \begin{pmatrix} \theta_{j,1}(z,x) \\ \theta_{j,2}(z,x) \end{pmatrix}, \quad \Phi_j(z,x) = \begin{pmatrix} \phi_{j,1}(z,x) \\ \phi_{j,2}(z,x) \end{pmatrix}, \quad U_{j,+}(z,x) = \begin{pmatrix} u_{j,+,1}(z,x) \\ u_{j,+,2}(z,x) \end{pmatrix}, \\ j = 1, 2, \ x \geq x_0, \tag{5.36}$$

the solutions associated with B_j, $j = 1, 2$, which are defined in (2.14b) and (2.17), we introduce

$$g_{j,k}(z,x) = \phi_{j,k}(z,x) u_{j,+,k}(\bar z, x)^*, \quad j,k \in \{1,2\}, \ x \geq x_0. \tag{5.37}$$

Using the asymptotic expansions (5.2)–(5.4) for $\Theta_j(z, x)$, $\Phi_j(z, x)$, and $U_{j,+}(z, x)$, and the analogous ones for $\Theta_j(\bar z, x)^*$, $\Phi_j(\bar z, x)^*$, and $U_{j,+}(\bar z, x)^*$, one verifies for each fixed $x > x_0$,

$$g_{j,k}(z,x) \underset{\substack{|z|\to\infty \\ z\in\mathbb{C}_+}}{=} (i/2) I_m + o(1), \quad j,k \in \{1,2\}, \tag{5.38}$$

assuming $B_j \in L^1([x_0, x_0 + R])^{2m\times 2m}$ for all $R > 0$, $j = 1, 2$, only. Next, using the fact that for each fixed $x > x_0$,

$$\phi_{1,k}(z,x)^{-1} \phi_{2,k}(z,x) \underset{\substack{|z|\to\infty \\ z\in\mathbb{C}_+}}{=} I_m + o(1), \quad k = 1, 2, \tag{5.39}$$

$$(u_{1,+,k}(\bar z,x)^*)^{-1} u_{2,+,k}(\bar z,x)^* \underset{\substack{|z|\to\infty \\ z\in\mathbb{C}_+}}{=} I_m + o(1), \quad k = 1, 2, \tag{5.40}$$

by (5.3), (5.4), one concludes

$$\begin{aligned} &\phi_{1,k}(z,x) u_{2,+,j}(\bar z,x)^* - u_{1,+,k}(z,x)\phi_{2,k}(\bar z,x)^* \\ &= \phi_{1,k}(z,x)\theta_{2,k}(\bar z,x)^* - \theta_{1,k}(z,x)\phi_{2,k}(\bar z,x)^* \\ &\quad + \phi_{1,k}(z,x)\big(M_{2,+}(z) - M_{1,+}(z)\big)\phi_{2,k}(\bar z,x)^* \underset{\substack{|z|\to\infty \\ z\in\mathbb{C}_+}}{=} o(1), \end{aligned} \tag{5.41}$$

using (5.38), (5.40), and $M_{2,+}(\bar z)^* = M_{2,+}(z)$. Combining hypothesis (5.33) and (5.3), one infers

$$\big\|\phi_{1,k}(z,x)\big(M_{2,+}(z) - M_{1,+}(z)\big)\phi_{2,k}(\bar z,x)^*\big\| \underset{\substack{|z|\to\infty \\ z\in\rho_{+,\ell}}}{=} o(1), \quad x \in (x_0, x_0 + a) \tag{5.42}$$

along the rays $\rho_{+,\ell}$, $\ell = 1, 2$. Thus, (5.41) implies

$$\big\|\phi_{1,k}(z,x)\theta_{2,k}(\bar z,x)^* - \theta_{1,k}(z,x)\phi_{2,k}(\bar z,x)^*\big\| \underset{\substack{|z|\to\infty\\ z\in\rho_{+,\ell}}}{=} o(1), \quad x \in (x_0, x_0+a) \tag{5.43}$$

along the rays $\rho_{+,\ell}$, $\ell = 1, 2$. The analogous estimate (5.43) holds along the complex conjugate rays $\bar\rho_{+,\ell}$, $\ell = 1, 2$, in the lower complex half-plane $\mathbb{C}_-$. To simplify notations we denote the open sector generated by $\rho_{+,1}$ and its complex conjugate $\bar\rho_{+,1}$ by $\mathcal{S}_1$, the open sector generated by the $\rho_{+,2}$ and its complex conjugate $\bar\rho_{+,2}$ by $\mathcal{S}_2$, the remaining sector in $\mathbb{C}_+$ is denoted by $\mathcal{S}_3$, and its complex conjugate sector in $\mathbb{C}_-$ is denoted by $\mathcal{S}_4$. Thus, one obtains a partition of $\mathbb{C}$ into

$$\mathbb{C} = \bigcup_{\ell=1}^{4} \overline{\mathcal{S}_\ell}, \tag{5.44}$$

where each sector $\mathcal{S}_\ell$, $1 \leq \ell \leq 4$, has opening angle strictly less than π. Since (each matrix element of) the expression under the norm in (5.43) is entire and of order less or equal to one, one can apply the Phragmén-Lindelöf principle (cf., e.g., [100, No. 322, p. 166–167, 379]) to each sector $\mathcal{S}_\ell$, $1 \leq \ell \leq 4$, and obtains that each matrix element under the norm in (5.43) is uniformly bounded in each sector and hence on all of $\mathbb{C}$. By Liouville's theorem, these matrix elements are all equal to certain constants. By the right-hand side of (5.43), these constants all vanish. Thus, we proved

$$\phi_{1,k}(z,x)\theta_{2,k}(\bar z,x)^* = \theta_{1,k}(z,x)\phi_{2,k}(\bar z,x)^* \text{ for all } x \in (x_0, x_0+a) \tag{5.45}$$

and hence

$$\phi_{1,k}(z,x)^{-1}\theta_{1,k}(z,x) = \theta_{2,k}(\bar z,x)^*(\phi_{2,k}(\bar z,x)^*)^{-1} \text{ for all } x \in (x_0, x_0+a). \tag{5.46}$$

Differentiating $\phi_{j,k}(z,x)^{-1}\theta_{j,k}(z,x)$, $j,k = 1,2$, with respect to x yields

$$\begin{aligned}
&\big(\phi_{j,1}(z,x)^{-1}\theta_{j,1}(z,x)\big)' \\
&= \phi_{j,1}(z,x)^{-1}((B_j)_{1,1}(x) - z)(\phi_{j,2}(z,x)\phi_{j,1}(z,x)^{-1}\theta_{j,1}(z,x) - \theta_{j,2}(z,x)), \quad (5.47)\\
&\big(\phi_{j,2}(z,x)^{-1}\theta_{j,2}(z,x)\big)' \\
&= \phi_{j,2}(z,x)^{-1}((B_j)_{1,1}(x) + z)(\phi_{j,1}(z,x)\phi_{j,2}(z,x)^{-1}\theta_{j,2}(z,x) - \theta_{j,1}(z,x)). \quad (5.48)
\end{aligned}$$

Multiplying (5.47) by $\phi_{j,1}(\bar z,x)^*(\phi_{j,1}(\bar z,x)^*)^{-1}$ and using (2.93), (2.95), and similarly, multiplying (5.48) by $\phi_{j,2}(\bar z,x)^*(\phi_{j,2}(\bar z,x)^*)^{-1}$ and using (2.94), (2.96) then yields

$$\big(\phi_{j,1}(z,x)^{-1}\theta_{j,1}(z,x)\big)' = \phi_{j,1}(z,x)^{-1}((B_j)_{1,1}(x) - z)(\phi_{j,1}(\bar z,x)^*)^{-1}, \tag{5.49}$$

$$\big(\phi_{j,2}(z,x)^{-1}\theta_{j,2}(z,x)\big)' = \phi_{j,2}(z,x)^{-1}((B_j)_{1,1}(x) + z)(\phi_{j,2}(\bar z,x)^*)^{-1}. \tag{5.50}$$

In exactly the same way one derives

$$\begin{aligned}
&\big(\theta_{j,1}(\bar z,x)^*(\phi_{j,1}(\bar z,x)^*)^{-1}\big)' \\
&= (\theta_{j,1}(\bar z,x)^*(\phi_{j,1}(\bar z,x)^*)^{-1}\phi_{j,2}(\bar z,x)^* - \theta_2(\bar z,x)^*)((B_j)_{1,1}(x) - z)(\phi_{j,1}(\bar z,x)^*)^{-1} \\
&= \phi_{j,1}(z,x)^{-1}((B_j)_{1,1}(x) - z)(\phi_{j,1}(\bar z,x)^*)^{-1}, \quad (5.51)\\
&\big(\theta_{j,2}(\bar z,x)^*(\phi_{j,2}(\bar z,x)^*)^{-1}\big)' \\
&= (\theta_{j,2}(\bar z,x)^*(\phi_{j,2}(\bar z,x)^*)^{-1}\phi_{j,1}(\bar z,x)^* - \theta_{j,1}(\bar z,x)^*)((B_j)_{1,1}(x) + z)(\phi_{j,2}(\bar z,x)^*)^{-1} \\
&= \phi_{j,2}(z,x)^{-1}((B_j)_{1,1}(x) + z)(\phi_{j,2}(\bar z,x)^*)^{-1}, \quad (5.52)
\end{aligned}$$

using (2.89)–(2.92). Thus, (5.46) implies

$$\phi_{1,1}(\bar z,x)^*((B_1)_{1,1}(x)-z)^{-1}\phi_{1,1}(z,x)=\phi_{2,1}(\bar z,x)^*((B_2)_{1,1}(x)-z)^{-1}\phi_{2,1}(z,x), \tag{5.53}$$

$$\phi_{1,2}(\bar z,x)^*((B_1)_{1,1}(x)+z)^{-1}\phi_{1,2}(z,x)=\phi_{2,2}(\bar z,x)^*((B_2)_{1,1}(x)+z)^{-1}\phi_{2,2}(z,x), \tag{5.54}$$

$$\theta_{1,1}(\bar z,x)^*((B_1)_{1,1}(x)-z)^{-1}\theta_{1,1}(z,x)=\theta_{2,1}(\bar z,x)^*((B_2)_{1,1}(x)-z)^{-1}\theta_{2,1}(z,x), \tag{5.55}$$

$$\theta_{1,2}(\bar z,x)^*((B_1)_{1,1}(x)+z)^{-1}\theta_{1,2}(z,x)=\theta_{2,2}(\bar z,x)^*((B_2)_{1,1}(x)+z)^{-1}\theta_{2,2}(z,x) \tag{5.56}$$

for a.e. $x\in(x_0,x_0+a)$. Thus far we only used $B_j\in L^1([x_0,x_0+R])^{2m\times 2m}$ for all $R>0$, $j=1,2$ and (5.35).

In the special case $m=1$, each of the equations (5.53)–(5.56) allows for the completion of the proof of (5.34). Indeed, using the fact that

$$\overline{\phi_{j,k}(\bar z,x)}=\phi_{j,k}(z,x),\quad \overline{\theta_{j,k}(\bar z,x)}=\theta_{j,k}(z,x),\quad j,k\in\{1,2\}, \tag{5.57}$$

and taking for instance (5.53), one infers for a.e. $x\in(x_0,x_0+a)$, that

$$\frac{\phi_{1,1}(z,x)^2}{\phi_{2,1}(z,x)^2}=\frac{(B_1)_{1,1}(x)-z}{(B_2)_{1,1}(x)-z}. \tag{5.58}$$

Since all zeros (and poles) of the left-hand side of (5.58) have even multiplicity, while all zeros (and poles) of the right-hand side of (5.57) are simple, one concludes, assuming only that $B_j\in L^1([x_0,x_0+R])^{2\times 2}$ for all $R>0$, $j=1,2$, that

$$(B_1)_{1,1}(x)=(B_2)_{1,1}(x)\ \text{ for a.e. } x\in(x_0,x_0+a). \tag{5.59}$$

Thus for the case $m=1$, we see by (5.53), and (5.54), (5.57), and (5.59), for a.e. $x\in(x_0,x_0+a)$, that

$$\phi_{1,k}^2(z,x)=\phi_{2,k}^2(z,x),\quad k=1,2. \tag{5.60}$$

Now, (2.92), (5.46), and (5.57) show, for a.e. $x\in(x_0,x_0+a)$, that

$$\begin{aligned}\phi_{1,1}(z,x)\phi_{1,2}(z,x)&=\frac{\phi_{1,1}(z,x)}{\theta_{1,1}(z,x)}-\frac{\phi_{1,2}(z,x)}{\theta_{1,2}(z,x)}\\&=\frac{\phi_{2,1}(z,x)}{\theta_{2,1}(z,x)}-\frac{\phi_{2,2}(z,x)}{\theta_{2,2}(z,x)}=\phi_{2,1}(z,x)\phi_{2,2}(z,x).\end{aligned} \tag{5.61}$$

By (2.2a) we see that

$$(\phi_{j,1}^2(z,x))'=2(z-(B_j(x))_{1,1})\phi_{j,1}(z,x)\phi_{j,2}(z,x)+(B_j(x))_{1,2}\phi_{j,1}^2(z,x),\quad j=1,2. \tag{5.62}$$

Thus, by (5.59), (5.60), and (5.61),

$$(B_1(x))_{1,2}=(B_2(x))_{1,2}\ \text{ for a.e. } x\in(x_0,x_0+a). \tag{5.63}$$

Together, (5.59) and (5.63) imply (5.34) in the special case $m=1$.

Unfortunately, the case $m>1$ appears to be quite a bit more involved. To deal with this case we first note that taking determinants in (5.53) yields

$$\frac{\det(\phi_{1,1}(\bar z,x,x_0,\alpha)^*)\det(\phi_{1,1}(z,x,x_0,\alpha))}{\det(\phi_{2,1}(\bar z,x,x_0,\alpha)^*)\det(\phi_{2,1}(z,x,x_0,\alpha)}=\frac{\det((B_1)_{1,1}(x)-zI_m))}{\det((B_2)_{1,1}(x)-zI_m))} \tag{5.64}$$

for a.e. $x \in (x_0, x_0 + a)$. Next, we intend to prove that

$$\det((B_1)_{1,1}(x) - zI_m)) = \det((B_2)_{1,1}(x) - zI_m)) \text{ for a.e. } x \in (x_0, x_0 + a). \quad (5.65)$$

Given the fact that $(B_j)_{1,1}(x)$, $j = 1, 2$, is self-adjoint, showing (5.65) is equivalent to showing that $B_1(x)$ and $B_2(x)$ are unitarily equivalent for a.e. $x \in (x_0, x_0 + a)$. Arguing by contradiction, we assume that at least one pair of eigenvalues of $B_1(x)$ and $B_2(x)$ differs. Thus, fixing $x_1 \in (x_0, x_0 + a)$, let $\lambda(x_1)$ be an eigenvalue of $B_1(x_1)$ but not of $B_2(x_1)$. Then (5.64) implies, for all $\alpha \in \mathbb{C}^{m\times 2m}$ satisfying (2.9), that

$$\det(\phi_{1,1}(\lambda(x_1), x_1, x_0, \alpha)) = 0. \quad (5.66)$$

Next, for $\lambda \in \mathbb{R}$ and $x > x_0$ define

$$\begin{aligned} N(\lambda, x, \alpha) =& \theta_{1,2}(\lambda, x, x_0, \alpha)\theta_{1,2}(\lambda, x, x_0, \alpha)^* \\ &+ \phi_{1,2}(\lambda, x, x_0, \alpha)\phi_{1,2}(\lambda, x, x_0, \alpha)^*. \end{aligned} \quad (5.67)$$

Then, $N(\lambda, x, \alpha)$ is strictly positive definite,

$$N(\lambda, x, \alpha) > 0. \quad (5.68)$$

Indeed, suppose $Nf = 0$ for some $f \in \mathbb{C}^m$, then

$$\theta_{1,2}(\lambda)\theta_{1,2}(\lambda)^* f + \phi_{1,2}(\lambda)\phi_{1,2}(\lambda)^* f = 0 \quad (5.69)$$

implies

$$\theta_{1,2}(\lambda)\theta_{1,2}(\lambda)^* f = 0, \ \phi_{1,2}(\lambda)\phi_{1,2}(\lambda)^* f = 0 \quad (5.70)$$

and hence

$$\theta_{1,2}(\lambda)^* f = 0, \ \phi_{1,2}(\lambda)^* f = 0. \quad (5.71)$$

Thus,

$$f = (\theta_{1,1}(\lambda)\phi_{1,2}(\lambda)^* - \phi_{1,1}(\lambda)\theta_{1,2}(\lambda)^*)f = 0 \quad (5.72)$$

by (2.96), and hence $f = 0$ proves (5.68). Introducing $\alpha_0 = (I_m \ 0) \in \mathbb{C}^{m\times 2m}$ and $\gamma = (\gamma_1 \ \gamma_2) \in \mathbb{C}^{m\times 2m}$ defined by

$$\begin{aligned} \gamma_1 = & [\theta_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)\theta_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)^* \quad (5.73) \\ & + \phi_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)\phi_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)^*]^{-1/2}\theta_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0), \\ \gamma_2 = & [\theta_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)\theta_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)^* \quad (5.74) \\ & + \phi_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)\phi_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)^*]^{-1/2}\phi_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0), \end{aligned}$$

one verifies $\gamma\gamma^* = I_m$ (by (5.73) and (5.74)) and $\gamma J \gamma^* = 0$ (by (2.94)). Thus, γ satisfies (2.9). Next, since

$$\phi_{1,1}(\lambda(x_1), x_1, x_0, \gamma) = \phi_{1,1}(\lambda(x_1), x_1, x_0, \alpha_0)\gamma_1^* - \theta_{1,1}(\lambda(x_1), x_1, x_0, \alpha_0)\gamma_2^* \quad (5.75)$$

as a special case of (2.97), one derives

$$\begin{aligned} \phi_{1,1}(\lambda(x_1), x_1, x_0, \gamma) = & [\phi_{1,1}(\lambda(x_1), x_1, x_0, \alpha_0)\theta_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)^* \\ & - \theta_{1,1}(\lambda(x_1), x_1, x_0, \alpha_0)\phi_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)^*] \times \\ & \times [\theta_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)\theta_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)^* \\ & + \phi_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)\phi_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)^*]^{-1/2} \\ = & -[\theta_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)\theta_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)^* \quad (5.76) \\ & + \phi_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)\phi_{1,2}(\lambda(x_1), x_1, x_0, \alpha_0)^*]^{-1/2} < 0. \end{aligned}$$

using (2.96). This contradiction to (5.66) proves (5.65). Hence for $\lambda \in \mathbb{R}$ and for a.e. $x \in (x_0, x_0 + a)$

$$|\det(\phi_{1,1}(\lambda, x, x_0, \alpha))| = |\det(\phi_{2,1}(\lambda, x, x_0, \alpha))|, \tag{5.77}$$

by (5.64). Equation (5.77) implies that for a.e. $x_1 \in (x_0, x_0+a)$, the family of Dirac operators $D_+(\alpha, \alpha_0)$ in $L^2([x_0, x_1])^{2m}$, defined by

$$\begin{aligned} D_+(\alpha, \alpha_0) &= J\frac{d}{dx} - B, \\ \operatorname{dom}(D_+(\alpha, \alpha_0)) &= \{\phi \in L^2([x_0, x_1])^{2m} \mid \phi \in \mathrm{AC}([x_0, x_1])^{2m}; \\ &\qquad \alpha\phi(x_0) = 0,\, \alpha_0\phi(x_1) = 0;\, (J\phi' - B\phi) \in L^2([x_0, x_1])^{2m}\}, \end{aligned} \tag{5.78}$$

with $\alpha_0 = (I_m\ 0)$, have identical spectra for all boundary data $\alpha \in \mathbb{C}^{m\times 2m}$ satisfying (2.9). Hence, assuming $B_j \in L^\infty([x_0, x_0 + a])^{2m\times 2m}$, $j = 1, 2$, one can apply Theorem 2.3 of Malamud [88] and obtains (5.34). ☐

We should note that Malamud's Theorem 2.3 in [88] only requires the equality of $m^2 + 1$ spectra (associated with linearly independent boundary data indexed by $\alpha \in \mathbb{C}^{m\times 2m}$) in order to conclude (5.34).

There is no particular significance of the rays ρ_ℓ, $\ell = 1, 2$, in Theorem 5.3. Any non-selfintersecting Jordan arc that tends to infinity in the sectors $\varepsilon \le \arg(z) \le (\pi/2) - \varepsilon$ and $(\pi/2) + \varepsilon \le \arg(z) \le \pi - \varepsilon$ for some $0 < \varepsilon < \pi/4$ will do.

Remark 5.4. We were not able to prove (5.34) directly from (5.53)–(5.56), without resorting to the arguments involving (5.77) and (5.78). To conclude the proof according to the Borg-type Theorem 2.3 of Malamud [88] (cf. also Theorem 4 in [89]), requires the introduction of the extra hypothesis $B_j \in L^\infty([x_0, x_0+a])^{2m\times 2m}$, $j = 1, 2$ in the matrix context $m > 1$, since the construction of transformation operators for Dirac-type systems, to date, uses such an additional hypothesis on B. This extra hypothesis is clearly superfluous in the case $m = 1$. Obviously, one conjectures that this extra hypothesis on B_j should also be redundant in Theorem 5.3, but this appears to require nontrivial future efforts. In this context it might be interesting to note that the higher-order expansions (5.8)–(5.10) do not determine B uniquely. An explicit analysis shows that while they do determine $B_{1,2}$, they only determine $B_{1,1}^2$, not $B_{1,1}$ itself. So that approach does not aide in proving (5.34) (besides, it would require the additional hypotheses (5.6) on B).

The corresponding local uniqueness result in terms of $M(z, x_0, \alpha)$ then reads as follows.

Theorem 5.5. *Fix $x_0 \in \mathbb{R}$ and suppose $A_j = I_{2m}$, $B_j \in L^1_{\mathrm{loc}}(\mathbb{R})^{2m\times 2m}$, and $B_j = B_j^*$ a.e. on $\mathbb{R}$, $j = 1, 2$. Suppose also that B_j is in the normal form given in* (5.1) *a.e. on (x_0, ∞), $j = 1, 2$. Denote by $M_j(z, x_0, \alpha)$, the unique Weyl-Titchmarsh matrices* (2.76) *corresponding to the Dirac-type operators D_j, $j = 1, 2$, in* (2.75)*. Then,*

$$\text{if for some } a > 0,\ B_1(x) = B_2(x) \ \text{ for a.e. } x \in (x_0 - a, x_0 + a), \tag{5.79}$$

one obtains

$$\|M_1(z, x_0, \alpha) - M_2(z, x_0, \alpha)\|_{\mathbb{C}^{2m\times 2m}} \underset{\substack{|z|\to\infty \\ z\in\rho_+}}{=} O\big(e^{-2\mathrm{Im}(z)a}\big) \tag{5.80}$$

along any ray $\rho_+ \subset \mathbb{C}_+$ with $0 < \arg(z) < \pi$ and for all $\alpha \in \mathbb{C}^{m\times 2m}$ satisfying (2.9)*. Conversely, fix a $\hat\alpha \in \mathbb{C}^{m\times 2m}$ satisfying* (2.9) *and if $m > 1$, assume in addition that $B_j \in L^\infty([x_0-a, x_0+a])^{2m\times 2m}$, $j = 1, 2$. Moreover, suppose that for all $\varepsilon > 0$,*

$$\|M_1(z,x_0,\hat\alpha_1) - M_2(z,x_0,\hat\alpha_1)\|_{\mathbb{C}^{2m\times 2m}} \underset{\substack{|z|\to\infty \\ z\in\rho_{+,\ell}}}{=} O\big(e^{-2\mathrm{Im}(z)(a-\varepsilon)}\big), \quad \ell = 1, 2, \tag{5.81}$$

along a ray $\rho_{+,1} \subset \mathbb{C}_+$ with $0 < \arg(z) < \pi/2$ and along a ray $\rho_{+,2} \subset \mathbb{C}_+$ with $\pi/2 < \arg(z) < \pi$. Then

$$B_1(x) = B_2(x) \text{ for a.e. } x \in [x_0 - a, x_0 + a]. \tag{5.82}$$

Proof. (5.80) is proved by combining (2.77), and (5.31), (5.32), and (5.82) then follows by combining (2.77), and (5.33), (5.34), taking into account the asymptotic expansions

$$M_\pm(z,x_0) \underset{|z|\to\infty}{=} \pm i I_m + o(1) \tag{5.83}$$

along any ray with $\varepsilon < \arg(z) < \pi - \varepsilon$ in the case of Dirac-type operators (cf. (3.1)). ☐

Remark 5.6. Theorem 5.3 and Theorem 5.5 yield new global uniqueness theorems for half-line and full-line Dirac-type operators, extending the classical Borg-Marchenko-type results. Indeed, if (5.33) (resp., (5.81)) holds for all $a > 0$, then (5.34) (resp. (5.82)) holds for a.e. $x \in [x_0, \infty)$ (resp., for a.e. $x \in \mathbb{R}$).

In the case of scalar Schrödinger operators, the analog of Theorem 5.3 is due to Simon [114]. An alternative proof, applicable to matrix-valued Schrödinger operators was presented in [50] (cf. also [41]). More recently, yet another proof was found by Bennewitz [13] (following some ideas in [16]). These results extend the classical (global) uniqueness results due to Borg [16] and Marchenko [91], [92] (cf. also [14]), which state that half-line m-functions uniquely determine the corresponding potential coefficient. The Dirac-type results presented in this section (especially, all local considerations) appear to be new, even in the special case $m = 1$. Previous results in the Dirac case focused on global uniqueness questions only. We refer to Gasymov and Levitan [34] in the case $m = 1$ and to Lesch and Malamud [81] in the matrix case $m \in \mathbb{N}$. Most recently, Alexander Sakhnovich kindly informed us that his integral representation of the Weyl-Titchmarsh matrix in [103] can be used to derive asymptotic expansions for the Weyl-Titchmarsh matrix and its associated matrix-valued spectral function, and also yields a result analogous to Theorem 5.3 (i) for a certain class of canonical systems. Moreover, in the case of skew-adjoint Dirac-type systems, similar results are discussed in [104] and applied to the nonlinear Schrödinger equation on a half-axis.

Although not directly used in this paper, it should be pointed out that inverse monodromy problems for canonical systems received a lot of attention (some of it very recently). The interested reader is referred to [4], [5], [6], [87], [88], [89], [109], [112] and the extensive literature cited therein. Moreover, inverse spectral theory associated with canonical systems is discussed in [96], [104], [106], [107], [109], [110], [111], [112] (see also the extensive literature cited in [41]).

6. Trace Formulas and Borg-Type Theorems

In our final section we derive a trace formula for B and then discuss its application to Borg-type uniqueness theorems for Dirac-type operators.

Theorem 6.1. *Assume Hypothesis 2.1 with* $A = I_{2m}$*, and let* $\alpha_0 = (I_m \ 0) \in \mathbb{C}^{m\times 2m}$*. Fix* $x_0 \in \mathbb{R}$ *and suppose that for all* $R > 0$*,*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{y\in[x_0,x_0+R]} \bigg\| \int_y^{x_0+R} dx'\, B(x')\exp(2iz(x'-y)) + \frac{1}{2iz}B(y)\bigg\|_{\mathbb{C}^{2m\times 2m}} \\
&+ \operatorname*{ess\,sup}_{y\in[x_0-R,x_0]} \bigg\| \int_{x_0-R}^{y} dx'\, B(x')\exp(2iz(x'-y)) - \frac{1}{2iz}B(y)\bigg\|_{\mathbb{C}^{2m\times 2m}} \\
&\underset{\substack{|z|\to\infty\\ z\in\rho_+}}{=} o\big(|z|^{-1}\big)
\end{aligned} \tag{6.1}$$

along a ray $\rho_+ \subset \mathbb{C}_+$*. In addition, assume* $B_{k,k'}B_{\ell,\ell'} \in L^1_{\text{loc}}(\mathbb{R})^{m\times m}$ *for all* $k,k',\ell,\ell' \in \{1,2\}$*. Then, with* $\Upsilon(\lambda,x,\alpha_0)$ *defined in* (2.83)*,*

$$\begin{aligned}
&\begin{pmatrix} B_{1,1}(x)-B_{2,2}(x) & B_{1,2}(x)+B_{2,1}(x) \\ B_{1,2}(x)+B_{2,1}(x) & B_{2,2}(x)-B_{1,1}(x)\end{pmatrix} \\
&= \lim_{\substack{|z|\to\infty\\ z\in\rho_+}} 2\int_{\mathbb{R}} d\lambda\, z^2(\lambda-z)^{-2}\Upsilon(\lambda,x,\alpha_0) \ \text{ for a.e. } x\in\mathbb{R}.
\end{aligned} \tag{6.2}$$

Proof. By (2.78),

$$\frac{d}{dz}\ln(M(z,x,\alpha_0)) = \int_{\mathbb{R}} d\lambda\,(\lambda-z)^{-2}\Upsilon(\lambda,x,\alpha_0). \tag{6.3}$$

Next, suppose that $x\in\mathbb{R}$ is a left and right Lebesgue point of B. By (4.65), (4.66) one obtains

$$\begin{aligned}
&\frac{d}{dz}\ln(M(z,x,\alpha_0)) \\
&\underset{\substack{|z|\to\infty\\ z\in\rho_+}}{=} \frac{1}{4}\begin{pmatrix} B_{1,1}(x+0)-B_{2,2}(x+0) & B_{1,2}(x+0)+B_{2,1}(x+0) \\ B_{1,2}(x+0)+B_{2,1}(x+0) & B_{2,2}(x+0)-B_{1,1}(x+0)\end{pmatrix} z^{-2} \\
&\quad + \frac{1}{4}\begin{pmatrix} B_{1,1}(x-0)-B_{2,2}(x-0) & B_{1,2}(x-0)+B_{2,1}(x-0) \\ B_{1,2}(x-0)+B_{2,1}(x-0) & B_{2,2}(x-0)-B_{1,1}(x-0)\end{pmatrix} z^{-2} + o\big(z^{-2}\big)
\end{aligned} \tag{6.4}$$

and hence

$$\begin{aligned}
&\frac{1}{2}\begin{pmatrix} B_{1,1}(x+0)-B_{2,2}(x+0) & B_{1,2}(x+0)+B_{2,1}(x+0) \\ B_{1,2}(x+0)+B_{2,1}(x+0) & B_{2,2}(x+0)-B_{1,1}(x+0)\end{pmatrix} \\
&+ \frac{1}{2}\begin{pmatrix} B_{1,1}(x-0)-B_{2,2}(x-0) & B_{1,2}(x-0)+B_{2,1}(x-0) \\ B_{1,2}(x-0)+B_{2,1}(x-0) & B_{2,2}(x-0)-B_{1,1}(x-0)\end{pmatrix} \\
&= \lim_{\substack{|z|\to\infty\\ z\in\rho_+}} 2\int_{\mathbb{R}} d\lambda\, z^2(\lambda-z)^{-2}\Upsilon(\lambda,x,\alpha_0).
\end{aligned} \tag{6.5}$$

Since a.e. $x\in\mathbb{R}$ is a Lebesgue point of B, one concludes (6.2). ☐

In the case $m = 1$, a trace formula for Dirac-type operators, using Krein spectral shift functions and exponential representations of Herglotz functions, was discussed in [116]. This circle of ideas was first introduced in connection with trace formulas of Schrödinger operators in [48] (see also [38], [39], [101], [102] in the scalar case $m = 1$. The corresponding case of trace formulas for matrix-valued Schrödinger operators was introduced in [37] (see also [21]).

Analogous trace formulas can be drived for all higher-order coefficients $M_k(x,\alpha_0)$ in (4.65) (see, e.g., [39] in connection with scalar Schrödinger operators).

A comparison of the trace formula (3.20) in [21] for Schrödinger operators with its Dirac-type counterpart (6.2) reveals characteristic differences. While in the Schrödinger case the trace formula directly involves the potential coefficient $Q(x)$, $M_1(x,\alpha_0)$ differs markedly from a constant multiple of $B(x)$, and consequently, the Dirac-type trace formula (6.2) does not directly involve $B(x)$ but certain linear combinations of $B_{j,k}(x)$. This is related to the fact that $M(z,x_0,\alpha_0)$ (or equivalently, $\Upsilon(\lambda,x_0,\alpha_0)$), in general, does not uniquely determine B a.e. In fact, there exists a typical ambiguity concerning the coefficients of D related to unitary gauge-transformations of D. In the case $m=1$ this ambiguity is well-known and discussed, e.g., in [34], [84, Sect. I.10], [85, Ch. 7]. These gauge transformations leave the spectrum of D invariant and suggest that we focus our attention on certain normal forms of D in connection with inverse spectral problems for Dirac-type operators.

Lemma 6.2. *Assume Hypothesis 2.17. Then $D=J\frac{d}{dx}-B$ is unitarily equivalent to $\widetilde D$, where $\widetilde D$ in $L^2(\mathbb{R})^{2m}$ is of the normal form*

$$\widetilde D = J\frac{d}{dx}-\widetilde B = \begin{pmatrix} -\widetilde B_{1,1} & -I_m\frac{d}{dx}-\widetilde B_{1,2} \\ I_m\frac{d}{dx}-\widetilde B_{1,2} & \widetilde B_{1,1}\end{pmatrix}. \tag{6.6}$$

Here $\widetilde B=\widetilde B^*$ a.e. and

$$\widetilde B_{1,1} = -(1/2)\mathrm{Im}\big(U_{1,1}^{-1}[(B_{1,2}+B_{2,1})-i(B_{1,1}-B_{2,2})]U_{2,2}\big)=\widetilde B_{1,1}^*, \tag{6.7}$$

$$\widetilde B_{1,2} = (1/2)\mathrm{Re}\big(U_{1,1}^{-1}[(B_{1,2}+B_{2,1})-i(B_{1,1}-B_{2,2})]U_{2,2}\big)=\widetilde B_{1,2}^*, \tag{6.8}$$

with $U_{j,j}\in\mathbb{C}^{m\times m}$, $j=1,2$, satisfying the first-order system of ordinary differential equations

$$iU_{j,j}'(x) = -(1/2)\big((-1)^j(B_{1,1}(x)+B_{2,2}(x))+i(B_{1,2}(x)-B_{2,1}(x))\big)U_{j,j}(x), \\ \text{for a.e. } x\in\mathbb{R}, \quad j=1,2. \tag{6.9}$$

Proof. We start with the unitary transformation V in $L^2(\mathbb{R})^{2m}$ defined by

$$V=\frac{1}{\sqrt2}\begin{pmatrix} iI_m & I_m \\ I_m & iI_m\end{pmatrix}, \quad V^{-1}=\frac{1}{\sqrt2}\begin{pmatrix} -iI_m & I_m \\ I_m & -iI_m\end{pmatrix}, \tag{6.10}$$

which maps D to D_1, where

$$D_1=V^{-1}DV = i\begin{pmatrix} I_m\frac{d}{dx} & 0 \\ 0 & -I_m\frac{d}{dx}\end{pmatrix} \\ -\frac12\begin{pmatrix} B_{1,1}+B_{2,2}-i(B_{1,2}-B_{2,1}) & B_{1,2}+B_{2,1}-i(B_{1,1}-B_{2,2}) \\ B_{1,2}+B_{2,1}+i(B_{1,1}-B_{2,2}) & B_{1,1}+B_{2,2}+i(B_{1,2}-B_{2,1})\end{pmatrix}. \tag{6.11}$$

Next, we introduce the unitary operator U in $L^2(\mathbb{R})^{2m}$ defined by

$$U=\begin{pmatrix} U_{1,1} & 0 \\ 0 & U_{2,2}\end{pmatrix}, \tag{6.12}$$

where the unitary $m\times m$ matrices $U_{j,j}\in\mathbb{C}^{m\times m}$ are solutions of the first-order system (6.9). Since by hypothesis $B_{j,k}\in L^1_{\mathrm{loc}}(\mathbb{R})^{m\times m}$, $1\le j,k\le 2$, the solutions of equation (6.9) are well-defined and $U_{j,j}\in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})^{m\times m}$, $j=1,2$. One computes

$$\widehat D = U^{-1}D_1U = \begin{pmatrix} iI_m\frac{d}{dx} & -\widehat B_{1,2} \\ -\widehat B_{1,2}^* & -iI_m\frac{d}{dx}\end{pmatrix}, \tag{6.13}$$

where $\widehat{B}_{1,2} \in L^1_{\text{loc}}(\mathbb{R})^{m\times m}$ and

$$\widehat{B}_{1,2}(x) = (1/2)U_{1,1}^{-1}(x)\big(B_{1,2}(x) + B_{2,1}(x) - i(B_{1,1}(x) - B_{2,2}(x)\big)U_{2,2}(x). \quad (6.14)$$

Finally, defining $\widetilde{D} = V\widehat{D}V^{-1}$, one arrives at (6.6)–(6.8). $\square$

Thus, unitary invariants of D (such as the spectrum, $\text{spec}(D)$, of D and its multiplicity) cannot determine B in general but at best a potential matrix of the type (normal form) $\widetilde{B}$ in (6.6). A further restriction on the solvability of inverse spectral problems for Dirac-type operators is mentioned in the following result.

Lemma 6.3. *Assume Hypotheses 2.17 and let $\omega = \omega^* \in \mathbb{C}^{m\times m}$ be a constant self-adjoint $m\times m$ matrix. Then $D = J\frac{d}{dx} - B$ is unitarily equivalent to $\widetilde{D}_\omega$ in $L^2(\mathbb{R})^{2m}$, where*

$$\widetilde{D}_\omega = J\frac{d}{dx} - \widetilde{B}_\omega = \begin{pmatrix} -\widetilde{B}_{\omega,1,1} & -I_m\frac{d}{dx} - \widetilde{B}_{\omega,1,2} \\ I_m\frac{d}{dx} - \widetilde{B}_{\omega,1,2} & \widetilde{B}_{\omega,1,1} \end{pmatrix}, \quad (6.15)$$

with

$$\widetilde{B}_{\omega,1,1} = -(1/2)\text{Im}\big(e^{i\omega}U_{1,1}^{-1}[(B_{1,2} + B_{2,1}) - i(B_{1,1} - B_{2,2})]U_{2,2}e^{i\omega}\big) = \widetilde{B}^*_{\omega,1,1},$$
$$\widetilde{B}_{\omega,1,2} = (1/2)\text{Re}\big(e^{i\omega}U_{1,1}^{-1}[(B_{1,2} + B_{2,1}) - i(B_{1,1} - B_{2,2})]U_{2,2}e^{i\omega}\big) = \widetilde{B}^*_{\omega,1,2}, \quad (6.16)$$

and with $U_{j,j}$, $j = 1, 2$, satisfying the first-order system (6.9).

Proof. Define

$$U_\omega = \begin{pmatrix} e^{i\omega} & 0 \\ 0 & e^{-i\omega} \end{pmatrix}. \quad (6.17)$$

Using the notation employed in the proof of Lemma 6.2 one verifies that

$$\widetilde{D}_\omega = VU_\omega(VU)^{-1}DVU(VU_\omega)^{-1}. \quad (6.18)$$

$\square$

In particular, choosing $\omega = (\pi/2)I_m$ effects the sign change $\widetilde{B} \to -\widetilde{B}$, with $\widetilde{B}$ given by (6.7), (6.8).

For detailed discussions of various normal forms for Dirac-type operators we refer to [34], [59], [84, Ch. 9], [85, Ch. 7] in the case $m = 1$ and to [33], [81], [88], [93, p. 193–195], [95] in the general matrix-valued case. Perhaps it should be noted that if D is in its normal form $\widetilde{D}$ as in (6.6), $\widetilde{D}^2$ turns into a $2m \times 2m$ matrix-valued Schrödinger operator under appropriate regularity assumptions on $\widetilde{B}$. Details on this fact and the relation between the M-matrices of $\widetilde{D}$ and $\widetilde{D}^2$ can be found in Section 3 of [41].

Next, we turn to Borg-type theorems, one of the principal topics of this paper. In 1946 Borg [15] proved, among a variety of other inverse spectral theorems, the following result.

Theorem 6.4 ([15])**.** *Assume $q \in L^2_{\text{loc}}(\mathbb{R})$ to be real-valued and periodic and let*

$$h = -d^2/dx^2 + q \quad (6.19)$$

be the associated self-adjoint Schrödinger operator in $L^2(\mathbb{R})$. Moreover, suppose that $\text{spec}(h) = [e_0, \infty)$ *for some $e_0 \in \mathbb{R}$. Then*

$$q(x) = e_0 \text{ for a.e. } x \in \mathbb{R}. \quad (6.20)$$

The analog of Theorem 6.4 for Dirac-type operators (in the case $m = 1$) was proven by Giacheti and Johnson [53] in 1984 (see also [35], [36], [47] in the special case where p is constant and [55] in the case where $p, q \in L^2(\mathbb{R})$ are real-valued and periodic).

Theorem 6.5 ([53])**.** *Assume $p, q \in L^\infty(\mathbb{R})$ to be real-valued and periodic and let*

$$d = \begin{pmatrix} -p & -\frac{d}{dx} - q \\ \frac{d}{dx} - q & p \end{pmatrix} \tag{6.21}$$

be the associated self-adjoint Dirac-type operator in $L^2(\mathbb{R})^2$. Moreover, suppose that $\mathrm{spec}(d) = \mathbb{R}$. *Then*

$$p(x) = q(x) = 0 \ \text{ for a.e. } x \in \mathbb{R}. \tag{6.22}$$

Traditionally, uniqueness results such as Theorems 6.4 and 6.5 are called Borg-type theorems. (However, this terminology is not uniquely adopted and hence a bit unfortunate. Indeed, inverse spectral results on finite intervals recovering the potential coefficient(s) from several spectra, were also pioneered by Borg in his celebrated paper [15], and hence are also coined Borg-type theorems in the literature, see, e.g., [86], [88], [89].)

A quick and natural proof of Theorem 6.4, based on a trace formula for q, was presented in [21]. This strategy of proof was then applied to the case of matrix-valued Schrödinger operators and the corresponding matrix-valued analog of Theorem 6.4 was also proved in [21] along these lines. A closer examination of the proof of Theorem 6.4 shows that periodicity of q is not the crucial element in the proof of the uniqueness result (6.20). The key ingredient (besides $\mathrm{spec}(h) = [e_0, \infty)$) is clearly the fact that for all $x \in \mathbb{R}$,

$$\xi(\lambda, x) = 1/2 \ \text{ for a.e. } \lambda \in \mathrm{spec}_{\mathrm{ess}}(h) \tag{6.23}$$

($\mathrm{spec}_{\mathrm{ess}}(\,\cdot\,)$ the essential spectrum), where $\xi(\cdot, x)$ is defined by

$$\xi(\lambda, x) = \lim_{\varepsilon \to 0} \pi^{-1} \mathrm{Im}(\ln(g(\lambda + i\varepsilon, x))) \ \text{ for a.e. } \lambda \in \mathbb{R}, \tag{6.24}$$

and $g(z, x)$ denotes Green's function (i.e., the integral kernel of the resolvent) of h on the diagonal,

$$g(z, x) = (h - z)^{-1}(x, x). \tag{6.25}$$

Completely analogous considerations apply to the Dirac-type case.

Real-valued periodic potentials are known to satisfy (6.23) but so are certain classes of real-valued quasi-periodic and almost-periodic potentials q (see, e.g., [23], [24], [25], [70], [72], [75], [76], [77], [78], [115]). In particular, the class of real-valued algebro-geometric finite-gap potentials q (a subclass of the set of real-valued quasi-periodic potentials) is a prime example satisfying (6.23) without necessarily being periodic. Traditionally, potentials q satisfying (6.23) are called *reflectionless* (see [24], [25], [77], [115]). Again the analogous notions apply to the Dirac-type case (cf., e.g., [23], [53], [71]).

Taking this circle of ideas as the point of departure for our derivation of Borg-type results for Dirac-type operators, we now use the reflectionless situation described in (6.23), actually, its proper analog for Dirac-type systems, as the model for the subsequent definition.

Definition 6.6. Assume Hypothesis 2.1 with $A = I_{2m}$, and let $\alpha_0 = (I_m \ 0) \in \mathbb{C}^{m\times 2m}$. Then B is called *reflectionless* if for all $x \in \mathbb{R}$,

$$\Upsilon(\lambda, x, \alpha_0) = (1/2)I_{2m} \text{ for a.e. } \lambda \in \operatorname{spec}_{\mathrm{ess}}(D). \tag{6.26}$$

Since hardly any confusion can arise, we will also call the Dirac-type operator D reflectionless if (6.26) is satisfied.

Given Definition 6.6, we turn to a Borg-type uniqueness theorem and formulate the analog of Theorem 6.4 for (reflectionless) Dirac-type operators.

Theorem 6.7. *Assume Hypothesis 2.1 with $A = I_{2m}$, and let $\alpha_0 = (I_m \ 0) \in \mathbb{C}^{m\times 2m}$. If for all $x \in \mathbb{R}$, $\Upsilon(\lambda, x, \alpha_0) = C$ is a constant $2m \times 2m$ matrix for a.e. $\lambda \in \mathbb{R}$, especially, if B is reflectionless and* $\operatorname{spec}(D) = \mathbb{R}$, *then*

$$B_{1,1}(x) = B_{2,2}(x), \quad B_{1,2}(x) = -B_{2,1}(x) \text{ for a.e. } x \in \mathbb{R}. \tag{6.27}$$

In particular, if D is assumed to be in its normal form (6.6)*, that is, of the type $\widetilde{D} = J\frac{d}{dx} - \widetilde{B}$, then*

$$\widetilde{B}(x) = 0 \text{ for a.e. } x \in \mathbb{R}. \tag{6.28}$$

Proof. The fact that $\int_{\mathbb{R}} d\lambda\,(\lambda - z)^{-2} = 0$ for all $z \in \mathbb{C}\backslash\mathbb{R}$, that a.e. $x \in \mathbb{R}$ is a Lebesgue point of B, and the trace formula (6.2), imply (6.27). Together with Lemma 6.2 this yields (6.28). $\square$

The analog of Theorem 6.7 for matrix-valued Schrödinger operators was recently proved in [21].

In the remainder of the section we will show that the case of periodic B is covered by Theorem 6.7 under appropriate uniform multiplicity assumptions on $\operatorname{spec}(D)$. In order to handle Floquet theoretic aspects of periodic Dirac-type operators D, we adopt the following assumptions until the end of this section.

Hypothesis 6.8. *In addition to Hypothesis 2.1 assume $A = I_{2m}$ and suppose that B is periodic, that is, there is an $\omega > 0$ such that $B(x + \omega) = B(x)$ for a.e. $x \in \mathbb{R}$.*

The following result has been proven in [21, Theorem 4.6].

Theorem 6.9 ([21], Theorem 4.6)**.** *Assume Hypothesis 6.8 and let $\alpha_0 = (I_m \ 0) \in \mathbb{C}^{2m\times m}$. If D has uniform spectral multiplicity $2m$, then for all $x \in \mathbb{R}$ and all $\lambda \in \operatorname{spec}(D)^o$,*

$$M_+(\lambda + i0, x, \alpha_0) = M_-(\lambda + i0, x, \alpha_0)^* = M_-(\lambda - i0, x, \alpha_0). \tag{6.29}$$

In particular, $M_-(z, x, \alpha_0)$ is the analytic continuation of $M_+(z, x, \alpha_0)$ (*and vice versa*) *through* $\operatorname{spec}(D)^o$*.*

Here A^o denotes the open interior of a set $A \subseteq \mathbb{R}$.

Strictly speaking, Theorem 4.6 in [21] was proved for matrix-valued Schrödinger operators. But the proof extends line by line to the corresponding Dirac-type situation and was predominantly formulated in terms of Hamiltonian systems notation (rather than Schrödinger operator specifics) in order to be applicable to the present context. In particular, the spectrum, $\operatorname{spec}(H)$, of the Schrödinger operator H should be replaced by that of D, the point spectrum, $\operatorname{spec}_p(H^D_{x_0})$, of the Dirichlet Schrödinger operator $H^D_{x_0}$ with a Dirichlet boundary condition at the point x_0 should simply be replaced by the set $\{\lambda \in \mathbb{R} \,|\, \det(\phi_1(\lambda, x_0 + \omega, x_0, \alpha_0)) = 0\}$, etc.

Theorem 6.10. *Suppose Hypothesis 6.8 and let $\alpha_0 = (I_m\ 0) \in \mathbb{C}^{2m\times m}$. If D has uniform spectral multiplicity $2m$, then D is reflectionless and for all $x \in \mathbb{R}$ and all $\lambda \in \operatorname{spec}(D)^o$,*

$$\Upsilon(\lambda, x, \alpha_0) = (1/2) I_{2m}. \tag{6.30}$$

Proof. This is clear from (2.77) and (6.29), which imply

$$M(\lambda + i0, x, \alpha_0) = -M(\lambda + i0, x, \alpha_0)^*. \tag{6.31}$$

□

Theorems 6.9 and 6.10 extend to more general situations (not necessarily periodic ones) as is clear from the corresponding results in [23], [53], [42], [75], [76], [77], [115] in the scalar case $m = 1$ (replacing the phrase "for all $\lambda \in \operatorname{spec}(D)^o$" by "for a.e. $\lambda \in \operatorname{spec}(D)^o$", etc.). For the corresponding matrix-valued Schrödinger operator case we refer to [78].

Corollary 6.11. *Assume Hypothesis 6.8. If D has uniform spectral multiplicity $2m$ and $\operatorname{spec}(D) = \mathbb{R}$, then*

$$B_{1,1}(x) = B_{2,2}(x), \quad B_{1,2}(x) = -B_{2,1}(x) \ \text{ for a.e. } x \in \mathbb{R}. \tag{6.32}$$

In particular, if D is assumed to be in its normal form $\widetilde{D} = J\frac{d}{dx} - \widetilde{B}$, with $\widetilde{B}$ given by (6.6)*, then*

$$\widetilde{B}(x) = 0 \ \text{ for a.e. } x \in \mathbb{R}. \tag{6.33}$$

Remark 6.12. The assumption of uniform (maximal) spectral multiplicity $2m$ in Corollary 6.11 is an essential one. Otherwise, one can easily construct nonconstant potentials B such that the associated operator D has overlapping band spectra and hence spectrum the whole real line. Also self-adjointness of B is crucial for Corollary 6.11 to hold (cf. the corresponding discussion in Remark 4.2 of [21] in the context of Schrödinger operators).

The analog of Corollary 6.11 for periodic matrix-valued Schrödinger operators was first proved by Depres [26] and recently rederived using such a trace formula approach in [21].

We note that all results presented in this paper also apply to matrix-valued finite-difference Hamiltonian systems. We refer the reader to [22] in this direction.

Finally, Borg-type uniqueness theorems for Hamiltonian systems are just a beginning. There is a natural extension of Borg's Theorem 6.4 to self-adjoint periodic Schrödinger, respectively, Dirac-type operators with one gap, respectively, two gaps in their spectrum. In the case of (scalar) Schrödinger operators, such an extension is due to Hochstadt [69] and the resulting potential q becomes twice the elliptic Weierstrass function. In the case of Dirac-type operators (with $m = 1$ and vanishing diagonal coefficients in B) such an extension involving elliptic functions can be found in [35], [36], [47] (see also [52]). Extensions to matrix-valued versions (i.e., for $m \geq 2$) are currently under active investigations.

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